1. (a) True: the orthogonal complement of the xy-plane is the subspace orthogonal to this plane, which is the z-axis.
   (b) False: the orthogonal complement of the row space is the nullspace, not the column space.
   (c) True: in general, $\dim(W) + \dim(W^\perp) = \dim(V)$, so here $\dim(W^\perp) = 9 - 4 = 5$.
   (d) True: as discussed in class, if we extend an orthonormal basis of W to an orthonormal basis of V, then the additional vectors form an orthonormal basis of $W^\perp$.
   (e) False: The vector $2\mathbf{e}_1 - \mathbf{e}_2$ is not in $W^\perp$, since the inner product of $2\mathbf{e}_1 - \mathbf{e}_2$ with $2\mathbf{e}_1 + \mathbf{e}_2$ is $(2)(2) + (-1)(1) = 3$ rather than 0. (In fact, $W^\perp$ is spanned by $\mathbf{e}_1 - 2\mathbf{e}_2$.)
   (f) True: since $\mathbf{v}$ is orthogonal to the basis vectors of W, it is orthogonal to every vector in W, so it is in $W^\perp$.
   (g) False: the projection of $\langle 1,2,3\rangle$ is $a_1 \langle 1,1,0\rangle$ where $a_1 = \langle 1,2,3\rangle \cdot \langle 1,1,0\rangle \,/\, \langle 1,1,0\rangle \cdot \langle 1,1,0\rangle = 3/2$, which yields $\langle 3/2, 3/2, 0\rangle$.
   (h) True: this is the Pythagorean theorem. Explicitly, $||\mathbf{v}||^2 = \left\langle \mathbf{w} + \mathbf{w}^\perp, \mathbf{w} + \mathbf{w}^\perp \right\rangle = \langle \mathbf{w}, \mathbf{w}\rangle + 2\left\langle \mathbf{w}, \mathbf{w}^\perp\right\rangle + \left\langle \mathbf{w}^\perp, \mathbf{w}^\perp\right\rangle = ||\mathbf{w}||^2 + \left|\left|\mathbf{w}^\perp\right|\right|^2$.
   (i) True: the closest vector is the orthogonal projection of $\langle 1,2,3\rangle$ into $x+y+z=0$. The plane is spanned by $(1,0,-1)$ and $(0,1,-1)$ so via the formula $M = A(A^TA)^{-1}A^T$ the projection matrix is $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ and the projection is this matrix times $\langle 1,2,3\rangle$, which is $\langle -1,0,1\rangle$.
   (j) False: the correct equation to solve is the normal equation $(A^TA)\hat{\mathbf{x}} = A^T\mathbf{c}$.
   (k) False: the orthogonal projection of a vector $\mathbf{w}^\perp$ in $W^\perp$ onto W is zero. (The orthogonal projection of a vector in W will be itself.)
   (l) True: as mentioned in class, projection matrices always have $P^2 = P$.
   (m) True: this set of vectors is the λ-eigenspace, and is a subspace.
   (n) False: for example, the real matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has nonreal eigenvalues.
   (o) False: we have seen many examples of matrices with repeated eigenvalues, such as the identity matrix.
   (p) True: since $\lambda = 0$ is a root of the characteristic polynomial, it is an eigenvalue.
   (q) False: since $\lambda = 1$ is a double root of the characteristic polynomial, the dimension of the eigenspace is at most 2, but it could be equal to 1.
   (r) False: we have $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$ which is usually not a multiple of $\mathbf{v}_1 + \mathbf{v}_2$.
   (s) True: as shown in class, eigenvectors with distinct eigenvalues are always linearly independent.
   (t) True: this is the diagonalizability criterion.
   (u) False: there are invertible matrices which are not diagonalizable, like $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
   (v) True: as mentioned in class, if there are n distinct eigenvalues, then each eigenvalue is a single root of the characteristic polynomial and all of the eigenspaces have dimension 1, so the matrix is diagonalizable.
   (w) False: matrices with repeated eigenvalues can still be diagonalizable, such as the identity matrix.
   (x) True: by the Cayley-Hamilton theorem, if we plug a matrix into its characteristic polynomial, the result is always the zero matrix.
   (y) True: this is part of the content of the spectral theorem (real symmetric matrices are orthogonally diagonalizable).
   (z) True: as discussed in class, $M^TM = I_n$ is equivalent to saying that M is an orthogonal matrix (whose columns are an orthonormal basis of $\mathbb{R}^n$).

2. For subspaces W of $\mathbb{R}^n$, we can find a basis of $W^\perp$ by finding the nullspace of the matrix whose rows are a basis for W.

   (a) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ row-reducing gives $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ so the nullspace (giving $W^\perp$) has basis $(1, -2, 1)$.

   (b) For $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$ row-reducing gives $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ so the nullspace (giving $W^\perp$) has basis $(1, -1, 0, 0), (0, 0, 1, -1)$.

   (c) The plane is the set of vectors (x, y, z) such that $2x + y + 3z = 0$ which is the nullspace of the matrix $[2\ 1\ 3]$. Therefore the orthogonal complement is the rowspace, which has basis $(2, 1, 3)$.

   (d) The orthogonal complement is 1-dimensional and contains $\mathbf{e}_3$, so $\{\mathbf{e}_3\}$ is a basis.

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3. This is the orthogonal complement of the vector $(1, 2, 3, 4)$, which is the nullspace of the matrix $[1\ 2\ 3\ 4]$. The nullspace has basis $(-2, 1, 0, 0)$, $(-3, 0, 1, 0)$, $(-4, 0, 0, 1)$.

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4. (a) Note that $\mathbf{w} = a_1\mathbf{w}_1$ where $a_1 = \langle \mathbf{v}, \mathbf{w}_1\rangle / \langle \mathbf{w}_1, \mathbf{w}_1\rangle = (6, 6, 6, 6) \cdot (1, 0, 1, 2)/(1, 0, 1, 2) \cdot (1, 0, 1, 2) = 24/6 = 4$, so $\mathbf{w} = (4, 0, 4, 8)$ and $\mathbf{w}^\perp = \mathbf{v} - \mathbf{w} = (2, 6, 2, -2)$.

   (b) The orthogonal projection is $a_1\mathbf{w}_1 + a_2\mathbf{w}_2$ where $a_1 = \langle \mathbf{v}, \mathbf{w}_1\rangle / \langle \mathbf{w}_1, \mathbf{w}_1\rangle = 4$ and $a_2 = \langle \mathbf{v}, \mathbf{w}_2\rangle / \langle \mathbf{w}_2, \mathbf{w}_2\rangle = 12/6 = 2$, yielding $4(1, 0, 1, 2) + 2(0, 1, 2, -1) = (4, 2, 8, 6)$.

   (c) This is the orthogonal projection of $\mathbf{v}$ into W_3 which is $a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3$ where $a_1 = \langle \mathbf{v}, \mathbf{w}_1\rangle / \langle \mathbf{w}_1, \mathbf{w}_1\rangle = 4$, $a_2 = \langle \mathbf{v}, \mathbf{w}_2\rangle / \langle \mathbf{w}_2, \mathbf{w}_2\rangle = 12/6 = 2$, and $a_3 = \langle \mathbf{v}, \mathbf{w}_3\rangle / \langle \mathbf{w}_3, \mathbf{w}_3\rangle = 0$ yielding $4(1, 0, 1, 2) + 2(0, 1, 2, -1) = (4, 2, 8, 6)$.

   (d) This is $A(A^TA)^{-1}A^T$ where $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$. The matrix product evaluates to $\dfrac{1}{6}\begin{bmatrix} 5 & -2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 5 \end{bmatrix}$.

   (e) This is the nullspace of the matrix $\begin{bmatrix} 0 & 1 & 2 & -1 \\ -2 & 1 & 0 & 1 \end{bmatrix}$ which row-reduces to $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$. The nullspace has basis $(1, 1, 0, 1)$, $(1, 2, -1, 0)$.

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5. To find the least-squares solution to a system $A\mathbf{x} = \mathbf{c}$, we solve the normal equation $(A^TA)\mathbf{x} = A^T\mathbf{c}$ to get $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{c}$.

   (a) With $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ we get $A^TA = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$ and $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1/5 \end{bmatrix}$.

   (b) With $\mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ we have $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$ so $A^TA = \begin{bmatrix} 9 & -1 \\ -1 & 3 \end{bmatrix}$ and $\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, corresponding to the line $y = 2x - 2$.

   (c) With $\mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ we have $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$ so $A^TA = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}$ and $\hat{\mathbf{x}} = \begin{bmatrix} 11/10 \\ 11/10 \end{bmatrix}$, corresponding to the line $y = 1.1x + 1.1$.

   (d) With $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ we have $A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$ so $A^TA = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 4 \end{bmatrix}$ and $\hat{\mathbf{x}} = \begin{bmatrix} 1/2 \\ 4/5 \\ 1/2 \end{bmatrix}$, corresponding to the parabola $y = 0.5x^2 + 0.8x + 0.5$.

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6. The matrix of orthogonal projection onto the column space of A is $P = A(A^TA)^{-1}A^T$.

   (a) We have $A = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ so with $A^TA = [14]$ we get $P = \frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$.

   (b) We have $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$ so with $A^TA = \begin{bmatrix} 6 & 6 \\ 6 & 9 \end{bmatrix}$ we get $P = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

   (c) The plane is the nullspace of $[1\ 2\ -1]$, and the nullspace has basis $(1,0,1)$, $(-2,1,0)$. With $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ we get $A^TA = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ and $P = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

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7. The eigenvalues are the roots of the characteristic polynomial $p(t) = \deg(tI_n - A)$, and the λ-eigenspace is the nullspace of $\lambda I_n - A$. Then A is diagonalizable when each eigenspace's dimension equals the multiplicity of the eigenvalue as a root of $p(t)$, and the diagonalizing matrix Q has columns equal to eigenvectors of A.

   (a) $p(t) = (t-5)(t+1)$, $\lambda = -1, 5$, basis of (-1)-eigenspace $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, basis of 5-eigenspace $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Diagonalizable with $Q = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

   (b) $p(t) = (t-1)^2$, $\lambda = 1, 1$, basis of 1-eigenspace $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Not diagonalizable.

   (c) $p(t) = (t-1)^3$, $\lambda = 1, 1, 1$, basis of 1-eigenspace $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Not diagonalizable.

   (d) $p(t) = (t-1)(t-2)(t-4)$, $\lambda = 1, 2, 4$, basis of 1-eigenspace $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, basis of 2-eigenspace $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, basis of 4-eigenspace $\begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$. Diagonalizable with $Q = \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

   (e) $p(t) = t(t+1)^2$, $\lambda = -1, -1, 0$, basis of (-1)-eigenspace $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, basis of 0-eigenspace $\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$. Diagonalizable with $Q = \begin{bmatrix} -2 & 1 & -2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   (f) $p(t) = t(t-1)^2$, $\lambda = 0, 1, 1$, basis of 0-eigenspace $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, basis of 1-eigenspace $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Diagonalizable with $Q = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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8. (a) Since the characteristic polynomial has degree 7, A is a 7×7 matrix.

   (b) The eigenvalues are 0 (multiplicity 4), 2 (multiplicity 2), and -4 (multiplicity 1).

   (c) The determinant is the product of the eigenvalues, which is $0^4 2^2 (-4)^1 = 0$, and the trace is the sum of the eigenvalues, which is $4 \cdot 0 + 2 \cdot 2 + 1 \cdot (-4) = 0$.

   (d) The 0-eigenspace has dimension 1, 2, 3, or 4, the 2-eigenspace has dimension 1 or 2, and the (-4)-eigenspace has dimension 1. A is diagonalizable precisely when the 0-eigenspace has dimension 4 and the 2-eigenspace has dimension 2.

(e) A diagonalization would be $\begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 2 & & \\ & & & & & 2 & \\ & & & & & & -4 \end{bmatrix}$ (all other entries not shown are zero).

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9. By the spectral theorem, to find an orthogonal diagonalization, we just compute an orthonormal basis for each eigenspace using Gram-Schmidt. (Note that for eigenspaces of dimension > 1, there are many possible choices of orthonormal basis.)

   (a) $p(t) = (t+3)(t+1)$, $\lambda = -3, -1$, orthonormal basis of (-3)-eigenspace $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, orthonormal basis of (-1)-eigenspace $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Orthogonal matrix $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$.

   (b) $p(t) = (t-1)(t-6)$, $\lambda = 1, 6$, orthonormal basis of 1-eigenspace $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, orthonormal basis of 6-eigenspace $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Orthogonal matrix $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

   (c) $p(t) = t^2(t-3)$, $\lambda = 0, 0, 3$, orthonormal basis of 0-eigenspace $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, orthonormal basis of 3-eigenspace $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Orthogonal matrix $Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

   (d) $p(t) = (t+4)(t+1)(t-2)$, $\lambda = -4, -1, 2$, orthonormal basis of (-4)-eigenspace $\frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$, orthonormal basis of (-1)-eigenspace $\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, orthonormal basis of 2-eigenspace $\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. Orthogonal matrix $Q = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

   (e) $p(t) = t(t-9)^2$, $\lambda = 0, 9, 9$, orthonormal basis of 0-eigenspace $\frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$, orthonormal basis of 9-eigenspace $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{18}}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$. Orthogonal matrix $Q = \begin{bmatrix} -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

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10. (a) We have $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A(\mathbf{v}_2 + \mathbf{v}_3) = 3\mathbf{v}_2 + 5\mathbf{v}_3$, and $A(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) = 4\mathbf{v}_1 - 3\mathbf{v}_2 + 15\mathbf{v}_3$.

    (b) Since A is 3×3 and the eigenvalues for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are distinct, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of 3 vectors in $\mathbb{R}^3$, so it is a basis. In terms of this basis we have the diagonalization $Q^{-1}AQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

    (c) For these vectors we have the change-of-basis matrix $Q = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ (the columns are the given vectors), in which case $A = QDQ^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & 8 & -14 \\ 3 & 9 & -10 \\ 3 & 6 & -7 \end{bmatrix}$.
