1. Identify each statement as either true or false, where the vector spaces U, V, W are finite-dimensional, the bases α, β, γ are ordered, and S, T are linear transformations.
   (a) If $T : V \to W$, then $T(\mathbf{0}_V) = \mathbf{0}_W$.
   (b) If $S, T : V \to W$ and S and T are equal on a basis of V, then $S(\mathbf{v}) = T(\mathbf{v})$ for all $\mathbf{v}$ in V.
   (c) $x^3 - x^2$ is in the kernel of the map $T : P_3(\mathbb{R}) \to \mathbb{R}^2$ with $T(p) = \langle p(0), p(1)\rangle$.
   (d) $\langle 2, 1, 2\rangle$ is in the image of the map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(a, b) = \langle a - b, a + b, a - b\rangle$.
   (e) If $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ with $T(1) = (1, 2)$ and $T(1 + x) = (2, 3)$, then $T(x)$ could be $(-1, 2)$.
   (f) If $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ with $T(1) = (1, 2)$ and $T(1 + x) = (2, 3)$, then $T(x^2)$ could be $(-1, 2)$.
   (g) There is a linear $T : \mathbb{R}^5 \to \mathbb{R}^4$ whose nullity is 2 and whose rank is 2.
   (h) There is a linear $T : \mathbb{R}^4 \to \mathbb{R}^5$ whose nullity is 2 and whose rank is 2.
   (i) If $T : V \to W$ is linear, then $\dim(\ker T) + \dim(\operatorname{im} T) = \dim(W)$.
   (j) If V is isomorphic to W, then $\dim(V) = \dim(W)$.
   (k) If $T : V \to V$ is invertible, then $[T^{-1}]_\beta^\beta = \left([T]_\beta^\beta\right)^{-1}$.
   (l) If $T : V \to V$, then $[T^2]_\alpha^\beta = \left([T]_\alpha^\beta\right)^2$.
   (m) For any $T : V \to V$, there always exists an invertible matrix Q such that $[T]_\beta^\beta = Q[T]_\alpha^\alpha Q^{-1}$.
   (n) If $S : V \to W$ and $T : U \to V$ are both linear, then $ST : U \to W$ is also linear.
   (o) If $S : V \to W$ and $T : U \to V$ are linear, then $[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$.
   (p) The change of basis matrix $[I]_\beta^\gamma$ is always invertible, and its inverse is $[I]_\gamma^\beta$.
   (q) If $T : V \to V$, then the matrices $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ are similar.
   (r) The pairing $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$ is an inner product on $P_{2021}(\mathbb{R})$.
   (s) The vector space $\mathbb{R}^2$ has exactly one inner product.
   (t) In any (real) inner product space, $\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$.
   (u) In any (real) inner product space, $|\langle \mathbf{v}, \mathbf{w}\rangle| \le ||\mathbf{v}||\,||\mathbf{w}||$.
   (v) In any (real) inner product space, $||\mathbf{v} + \mathbf{w}|| = ||\mathbf{v}|| + ||\mathbf{w}||$.
   (w) $(1, 1, -2, 0)$, $(1, -5, -2, 0)$, $(2, 0, 1, 7)$ is an orthogonal set in $\mathbb{R}^4$, with the standard dot product.
   (x) $\frac{1}{3}(1, 2, 2)$, $\frac{1}{\sqrt{2}}(0, 1, -1)$, $\frac{1}{\sqrt{18}}(-4, 1, 1)$ is an orthonormal basis of $\mathbb{R}^3$, with the standard dot product.
   (y) Every finite-dimensional inner product space has an orthogonal basis.
   (z) If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is an orthonormal basis for V, then $||\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4|| = 2$.

---

2. For each map $T : V \to W$, determine whether or not T is linear. If so show it, if not explain clearly why not.
   (a) $V = \mathbb{R}^4$, $W = \mathbb{R}^2$, $T(a, b, c, d) = (a + b + 1,\ c + d + 1)$.
   (b) $V = \mathbb{R}^4$, $W = \mathbb{R}^2$, $T(a, b, c, d) = (a + b,\ c + d)$.
   (c) $V = W = M_{2\times2}(\mathbb{R})$, $T(A) = 3A - 2A^T$.
   (d) $V = W = P_3(\mathbb{R})$, $T(p) = p''(x)$.
   (e) $V = W = M_{2\times2}(\mathbb{R})$, $T(A) = Q^{-1}AQ$, for a given invertible 2×2 matrix Q.
   (f) $V = W = M_{2\times2}(\mathbb{R})$, $T(A) = A^{-1}QA$, for a given invertible 2×2 matrix Q.

---

3. Suppose $V = P_2(\mathbb{R})$ and that $T : V \to V$ is linear with $T(1) = 1 - x^2$, $T(x) = 2x - x^2$, and $T(x^2) = 3 + x - x^2$.
   (a) Find $T(1 - 2x + x^2)$.
   (b) If β is the ordered basis $\beta = \{1, x, x^2\}$, find $[T]_\beta^\beta$.
   (c) If γ is the ordered basis $\gamma = \{2x^2, 1, 4x\}$, find $[T]_\gamma^\gamma$.

---

4. Suppose $V = P_2(\mathbb{R})$ with bases $\beta = \{2 - x, 3x^2, 1\}$ and $\gamma = \{1, x, x^2\}$.

   (a) Find the coordinate vectors $[1 + x - x^2]_\beta$ and $[1 + x - x^2]_\gamma$.

   (b) Find the change-of-basis matrices $[I]_\beta^\gamma$ and $[I]_\gamma^\beta$.

   (c) Suppose that $T : V \to V$ has $[T]_\beta^\beta = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 3 & 1 & 2 \end{bmatrix}$. Find $[T]_\gamma^\gamma$.

---

5. For each linear transformation $T : V \to W$, find bases for the kernel and for the image of T and verify the nullity-rank theorem. Then, using the given bases β for V and γ for W, find $[T]_\beta^\gamma$.

   (a) $V = W = P_2(\mathbb{R})$, $T(p) = xp'(x)$, $\beta = \gamma = \{1, x, x^2\}$.

   (b) $V = W = \mathbb{R}^4$,$T(a, b, c, d) = (a - b, b - c, c - d, d - a)$, $\beta = \{\langle 1, 1, 1, 1\rangle, \langle 2, 2, 0, 0\rangle, \langle 3, 0, 0, 0\rangle, \langle 0, 0, 0, 4\rangle\}$, $\gamma = \{\langle 1, 0, 0, 0\rangle, \langle 0, 1, 0, 0\rangle, \langle 0, 0, 1, 0\rangle, \langle 0, 0, 0, 1\rangle\}$.

   (c) $V = W = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$, $\beta = \gamma = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

   (d) $V = P_3(\mathbb{R})$, $W = M_{2\times 2}(\mathbb{R})$, $T(p) = \begin{bmatrix} p(0) & 0 \\ p(2) & 0 \end{bmatrix}$, $\beta = \{1, x, x^2, x^3\}$, $\gamma = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.

---

6. Find the associated matrix for each transformation of $\mathbb{R}^2$ with respect to the standard basis $\{(1, 0), (0, 1)\}$:

   (a) Triples the x-coordinate, halves the y-coordinate.

   (b) Rotates by $3\pi/4$ radians counterclockwise.

   (c) Projects onto $y = 3x$.

   (d) Projects onto the line spanned by $\langle 2, 3\rangle$.

   (e) Reflects across $y = 4x$.

   (f) Projects onto $y = 3x$, then projects onto $y = x$.

   (g) Scales by a factor of 4, then rotates by $2\pi/3$ radians counterclockwise, then reflects across $y = x$.

   (h) Reflects across $y = 2x$, then rotates by $\pi/2$ radians clockwise, then reflects across $y = 2x$ again.

---

7. In each given inner product space V, find $\langle \mathbf{v}, \mathbf{w}\rangle$, $||\mathbf{v}||$, $||\mathbf{w}||$, and the angle between $\mathbf{v}$ and $\mathbf{w}$:

   (a) $V = \mathbb{R}^3$ under the standard dot product, $\mathbf{v} = \langle 1, 3, 2\rangle$, $\mathbf{w} = \langle -5, 1, 1\rangle$.

   (b) $V = \mathbb{R}^4$ under the standard dot product, $\mathbf{v} = \langle 1, 1, 1, 1\rangle$, $\mathbf{w} = \langle -2, 0, 3, 6\rangle$.

   (c) $V = P_{25}(\mathbb{R})$ under the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$, $\mathbf{v} = x$, $\mathbf{w} = 1 - x$.

   (d) $V = M_{2\times 2}(\mathbb{R})$ under the inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$, $\mathbf{v} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$.

---

8. Apply the Gram-Schmidt procedure to each set S of vectors in the inner product space V to construct (i) an orthogonal set, and (ii) an orthonormal set with the same span as S:

   (a) $S = \{\langle 3, 4\rangle, \langle 1, 1\rangle\}$ inside $V = \mathbb{R}^2$ under the standard dot product.

   (b) $S = \{\langle 1, 1, 1\rangle, \langle 2, 3, 4\rangle\}$ inside $V = \mathbb{R}^3$ under the standard dot product.

   (c) $S = \{\langle 2, 1, 2\rangle, \langle 1, 1, 3\rangle, \langle -3, 5, 5\rangle\}$ inside $V = \mathbb{R}^3$ under the standard dot product.

   (d) $S = \{2, x\}$ inside $V = P_1(\mathbb{R})$ under the inner product $\langle f, g\rangle = \int_0^2 f(x)g(x)\,dx$.

   (e) $S = \{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\}$ inside $V = M_{2\times 2}(\mathbb{R})$ under the inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$.

---

9. Find the QR factorization of each matrix:

   (a) $\begin{bmatrix} -3 & 5 \\ 4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 4 \\ 6 & 9 \\ 3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$
