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1. (a) True: all linear transformations must send the zero vector of V to the zero vector of W.
   (b) True: a linear transformation is uniquely determined by its values on a basis.
   (c) True: for $p = x^3 - x^2$, $p(0) = 0^3 - 0^2 = 0$ and $p(1) = 1^3 - 1^2 = 0$, so $T(p) = \langle 0, 0\rangle$.
   (d) True: the system $\langle a-b, a+b, a-b\rangle = \langle 2, 1, 2\rangle$ has a solution $a = 3/2$, $b = -1/2$.
   (e) False: here, $T(x) = T(1+x) - T(1) = (2,3) - (1,2) = (1,1)$.
   (f) True: since $\{1, 1+x, x^2\}$ is a basis of $P_2(\mathbb{R})$, we can choose the values of T on these elements arbitrarily.
   (g) False: by nullity-rank, the nullity plus the rank equals the dimension of V, but $2+2 \neq 5$.
   (h) True: the nullity-rank theorem works out, and $T(a,b,c,d) = \langle a, b, 0, 0, 0\rangle$ is an explicit example.
   (i) False: the correct statement is $\dim(\ker T) + \dim(\operatorname{im} T) = \dim(V)$.
   (j) True: as shown in class, two vector spaces are isomorphic if and only if they have equal dimensions.
   (k) True: in general $[T^{-1}]_\gamma^\beta = ([T]_\gamma^\beta)^{-1}$ and this is the special case where $\gamma = \beta$.
   (l) False: a correct similar statement would be $[T^2]_\alpha^\beta = [T]_\beta^\beta [T]_\alpha^\beta$.
   (m) True: we can take Q to be the change-of-basis matrix $Q = [I]_\alpha^\beta$.
   (n) True: as noted in class, if $T : U \to V$ and $S : V \to W$ are linear, then so is $ST : U \to W$.
   (o) True: this is the correct composition formula.
   (p) True: as noted in class, the inverse of the change-of-basis matrix $[I]_\beta^\gamma$ is indeed $[I]_\gamma^\beta$.
   (q) True: we have $[T]_\beta^\beta = Q[T]_\alpha^\alpha Q^{-1}$ where $Q = [I]_\alpha^\beta$ is the change-of-basis matrix.
   (r) True: $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ is an inner product, as mentioned in class.
   (s) False: as we have seen, there are several inner products on $\mathbb{R}^2$.
   (t) True: this is axiom [I2].
   (u) True: this is the Cauchy-Schwarz inequality, which was shown in class.
   (v) False: in general $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$, but equality can only happen if $\{\mathbf{v}, \mathbf{w}\}$ is dependent.
   (w) True: these vectors are all orthogonal to one another.
   (x) True: these vectors are all orthogonal to one another, they all have length 1, and there are 3 vectors.
   (y) True: we can construct an orthogonal basis by applying Gram-Schmidt.
   (z) True: in general, $||a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4|| = \sqrt{a^2+b^2+c^2+d^2}$.

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2. In each case, we must check the properties [T1] and [T2]. A quick shortcut is also to check whether $T(\mathbf{0}) = \mathbf{0}$.

   (a) Not linear: neither [T1] nor [T2] holds since for example $T(ra, rb, rc, rd) = (ra+rb+1, rc+rd+1)$ is not equal to $rT(a,b,c,d) = (ra+rb+r, rc+rd+r)$. Also, $T(\mathbf{0}) = (1,1) \neq \mathbf{0}$.
   (b) Linear: [T1] $T(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2) = (a_1+a_2+b_1+b_2, c_1+c_2+d_1+d_2) = (a_1+b_1, c_1+d_1) + (a_2+b_2, c_2+d_2) = T(a_1,b_1,c_1,d_1) + T(a_2,b_2,c_2,d_2)$and [T2] $T(ra,rb,rc,rd) = (ra+rb, rc+rd) = r(a+b, c+d) = rT(a,b,c,d)$.
   (c) Linear: [T1] $T(A+B) = 3(A+B) - 2(A+B)^T = (3A - 2A^T) + (3B - 2B^T) = T(A) + T(B)$ and [T2] $T(rA) = 3rA - 2(rA)^T = r(3A - 2A^T) = rT(A)$.
   (d) Linear: [T1] $T(p+q) = (p+q)''(x) = p''(x) + q''(x) = T(p) + T(Q)$ and [T2] $T(cp) = (cp)''(x) = cp''(x) = cT(p)$.
   (e) Linear: [T1] $T(A+B) = Q^{-1}(A+B)Q = Q^{-1}AQ + Q^{-1}BQ = T(A) + T(B)$ and [T2] $T(rA) = Q^{-1}rAQ = r(Q^{-1}AQ) = rT(A)$.
   (f) Not linear: neither [T1] nor [T2] holds since for example $T(rA) = (rA)^{-1}Q(rA) = A^{-1}QA = T(A)$ is not equal to $rT(A)$. In fact, T is not even defined on the zero matrix.

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3. (a) $T(1-2x+x^2) = T(1) - 2T(x) + T(x^2) = (1-x^2) + 2(2x-x^2) + (3+x-x^2) = 4+5x-4x^2$.

(b) $[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ -1 & -1 & -1 \end{bmatrix}$ and $[T]_\gamma^\gamma = \begin{bmatrix} -1 & 1/2 & -2 \\ 6 & 1 & 0 \\ 1/2 & 0 & 2 \end{bmatrix}$.

(c) For $[T]_\gamma^\gamma$ note $T(2x^2) = 6+2x-2x^2 = -1(2x^2) + 6(1) + \frac{1}{2}(4x)$, $T(1) = 1-x^2 = \frac{1}{2}(2x^2) + 1(1) + 0(4x)$, $T(4x) = 8x - 4x^2 = -2(2x^2) + 0(1) + 2(4x)$. So matrix is as above.

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4. (a) Note $1+x-x^2 = -1(2-x) - \frac{1}{3}(3x^2) + 3(1)$ so $[1+x-x^2]_\beta = (-1, -\frac{1}{3}, 3)$ and $[1+x-x^2]_\gamma = (1,1,-1)$.

(b) Note that the columns of $[I]_\beta^\gamma$ are the vectors in the basis β expressed in terms of γ, so $[I]_\beta^\gamma = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$. Then $[I]_\gamma^\beta = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1/3 \\ 1 & 2 & 0 \end{bmatrix}$ either by computing the inverse matrix or by noting that $1 = 0(2-x) + 0(3x^2) + 1(1)$, $x = -1(2-x) + 0(3x^2) + 2(1)$, $x^2 = 0(2-x) + 1/3(3x^2) + 0(1)$.

(c) We have $[T]_\gamma^\gamma = [I]_\beta^\gamma [T]_\beta^\beta [I]_\gamma^\beta = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1/3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5/3 \\ -1 & -1 & -2/3 \\ 9 & 18 & 3 \end{bmatrix}$.

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5. Note that there are often many different choices for the bases of the kernel and image.

(a) $T(a+bx+cx^2) = bx + 2cx^2$, so $\ker(T) = \{a\}$ with basis $\{1\}$.
$\mathrm{im}(T) = \{bx + 2cx^2\}$ with basis $\{x, 2x^2\}$.
Nullity is 1, rank is 2, $\dim(P_2) = 3$, and $1+2=3$.

$T(1) = 0$, $T(x) = x$, $T(x^2) = 2x^2$ so $[T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) $\ker(T) = \{(a,a,a,a)\}$ with basis $\{(1,1,1,1)\}$.
$\mathrm{im}(T) = \{(a-b, b-c, c-d, d-a)\}$ with basis $\{(1,0,0,-1), (-1,1,0,0), (0,-1,1,0)\}$.
Nullity is 1, rank is 3, $\dim(\mathbb{R}^4) = 4$, and $1+3=4$.
$T(1,1,1,1) = (0,0,0,0)$, $T(2,2,0,0) = (0,2,0,-2)$, $T(3,0,0,0) = (0,-3,3,0)$, $T(0,0,0,4) = (0,0,-4,4)$

so $[T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & -2 & 0 & 4 \end{bmatrix}$.

(c) $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$ so $\ker(T) = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$ with basis $\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\}$.
$\mathrm{im}(T) = \{\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}\}$ with basis $\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\}$.
Nullity is 2, rank is 2, $\dim(M_{2\times 2}) = 4$, and $2+2=4$.
$T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$,

so $[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(d) $T(a+bx+cx^2+dx^3) = \begin{bmatrix} a & 0 \\ a+2b+4c+8d & 0 \end{bmatrix}$.
$\ker(T) = \{(-2c-4d)x + cx^2 + dx^3\}$ with basis $\{-2x+x^2, -4x+x^3\}$.
$\mathrm{im}(T) = \{\begin{bmatrix} a & 0 \\ a+2b+4c+8d & 0 \end{bmatrix}\}$ with basis $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$.
Nullity is 2, rank is 2, $\dim(P_3) = 4$, and $2+2=4$.

$T(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $T(x) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, $T(x^2) = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$, $T(x^3) = \begin{bmatrix} 0 & 0 \\ 8 & 0 \end{bmatrix}$, so $[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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6. (a) $\begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}$

   (b) $\begin{bmatrix} \cos(3\pi/4) & -\sin(3\pi/4) \\ \sin(3\pi/4) & \cos(3\pi/4) \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$

   (c) $\frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

   (d) $\frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$.

   (e) $\frac{1}{17}\begin{bmatrix} -15 & 8 \\ 8 & 15 \end{bmatrix}$

   (f) $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/5 \\ 1/5 & 3/5 \end{bmatrix}$

   (g) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & -2 \\ -2 & -2\sqrt{3} \end{bmatrix}$

   (h) $\frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} \cos(-\pi/2) & -\sin(-\pi/2) \\ \sin(-\pi/2) & \cos(-\pi/2) \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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7. (a) $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, $||\mathbf{v}|| = \sqrt{14}$, $||\mathbf{w}|| = \sqrt{27}$, $\theta = \cos^{-1}(\frac{0}{\sqrt{14}\sqrt{27}}) = \pi/2$.

   (b) $\langle \mathbf{v}, \mathbf{w} \rangle = 7$, $||\mathbf{v}|| = 2$, $||\mathbf{w}|| = 7$, $\theta = \cos^{-1}(\frac{7}{2 \cdot 7}) = \pi/3$.

   (c) $\langle \mathbf{v}, \mathbf{w} \rangle = -1/6$, $||\mathbf{v}|| = \sqrt{1/3}$, $||\mathbf{w}|| = \sqrt{1/3}$, $\theta = \cos^{-1}(\frac{-1/6}{\sqrt{1/3} \cdot \sqrt{1/3}}) = 2\pi/3$.

   (d) $\langle \mathbf{v}, \mathbf{w} \rangle = 8$, $||\mathbf{v}|| = \sqrt{30}$, $||\mathbf{w}|| = 3$, $\theta = \cos^{-1}(\frac{8}{\sqrt{30} \cdot 3})$.

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8. (a) $\mathbf{w}_1 = \mathbf{v}_1 = (3,4)$

   $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{v}_1$ where $a_1 = \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \frac{7}{25}$ so $\mathbf{w}_2 = (1,1) - \frac{7}{25}(3,4) = (\frac{4}{25}, -\frac{3}{25})$.

   Orthogonal set $\{(3,4), (\frac{4}{25}, -\frac{3}{25})\}$ yielding orthonormal set $\{\frac{1}{5}(3,4), \frac{1}{5}(4,-3)\}$.

   (b) $\mathbf{w}_1 = \mathbf{v}_1 = (1,1,1)$

   $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{v}_1$ where $a_1 = \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \frac{9}{3} = 3$ so $\mathbf{w}_2 = (2,3,4) - 3(1,1,1) = (-1,0,1)$.

   Orthogonal set $\{(1,1,1), (1,0,-1)\}$ yielding orthonormal set $\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,0,-1)\}$.

   (c) $\mathbf{w}_1 = \mathbf{v}_1 = (2,1,2)$,

   $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{v}_1$ where $a_1 = \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \frac{9}{9} = 1$ so $\mathbf{w}_2 = (1,1,3) - (2,1,2) = (-1,0,1)$

   $\mathbf{w}_3 = \mathbf{v}_2 - b_1\mathbf{v}_1 - b_2\mathbf{v}_2$ where $b_1 = \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \frac{9}{9} = 1$ and $b_2 = \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} = \frac{8}{2} = 4$ so $\mathbf{w}_2 = (-3,5,5) - (2,1,2) - 4(-1,0,1) = (-1,4,-1)$.

   Orthogonal set $\{(2,1,2), (-1,0,1), (-1,4,-1)\}$ yielding orthonormal set $\{\frac{1}{3}(2,1,2), \frac{1}{\sqrt{2}}(-1,0,1), \frac{1}{\sqrt{18}}(-1,4,-1)\}$.

   (d) $\mathbf{w}_1 = \mathbf{v}_1 = 2$

   $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{v}_1$ where $a_1 = \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \frac{\int_0^2 2x\,dx}{\int_0^2 2^2\,dx} = \frac{1}{2}$ so $\mathbf{w}_2 = x - 1$.

   Orthogonal set $\{2, x-1\}$.

   To make orthonormal, compute $||2|| = \sqrt{\int_0^2 2^2\,dx} = \sqrt{8}$, $||x-1|| = \sqrt{\int_0^2 (x-1)^2\,dx} = \sqrt{\int_0^2 (x^2-2x+1)\,dx} = \sqrt{2/3}$ yielding orthonormal set $\{\frac{2}{\sqrt{8}}, \frac{x-1}{\sqrt{2/3}}\}$.

(e) $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{v}_1$ where $a_1 = \dfrac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle} = \dfrac{12}{4} = 3$ so $\mathbf{w}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & 3 \end{bmatrix}$.

Orthogonal set $\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 0 & 3 \end{bmatrix}\}$.

To make orthonormal, compute $\left\|\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\| = 2$, $\left\|\begin{bmatrix} -2 & -1 \\ 0 & 3 \end{bmatrix}\right\| = \sqrt{14}$ yielding orthonormal set $\{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{\sqrt{14}}\begin{bmatrix} -2 & -1 \\ 0 & 3 \end{bmatrix}\}$.

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9. To find the QR factorization, apply Gram-Schmidt to the column vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. Then Q is given by the resulting orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ and R is upper-triangular with entries given by the inner products $\langle \mathbf{e}_i, \mathbf{v}_j\rangle$.

(a) For $\mathbf{v}_1 = (-3, 4)$, $\mathbf{v}_2 = (5, 2)$ get $\mathbf{e}_1 = \frac{1}{5}(-3, 4)$, $\mathbf{e}_2 = \frac{1}{5}(4, -3)$ and so $r_{1,1} = \langle \mathbf{e}_1, \mathbf{v}_1\rangle = 5$, $r_{1,2} = \langle \mathbf{e}_1, \mathbf{v}_2\rangle = -\frac{7}{5}$, $r_{2,2} = \langle \mathbf{e}_2, \mathbf{v}_2\rangle = \frac{26}{5}$. So $Q = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$, $R = \begin{bmatrix} 5 & -7/5 \\ 0 & 26/5 \end{bmatrix}$.

(b) For $\mathbf{v}_1 = (1, 3)$, $\mathbf{v}_2 = (2, 4)$ get $\mathbf{e}_1 = \frac{1}{\sqrt{10}}(1, 3)$, $\mathbf{e}_2 = \frac{1}{\sqrt{10}}(3, -1)$ and so $r_{1,1} = \langle \mathbf{e}_1, \mathbf{v}_1\rangle = \sqrt{10}$, $r_{1,2} = \langle \mathbf{e}_1, \mathbf{v}_2\rangle = 14/\sqrt{10}$, $r_{2,2} = \langle \mathbf{e}_2, \mathbf{v}_2\rangle = 2/\sqrt{10}$. So $Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, $R = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 2 \end{bmatrix}$.

(c) For $\mathbf{v}_1 = (-2, 6, 3)$, $\mathbf{v}_2 = (4, 9, 1)$ get $\mathbf{e}_1 = \frac{1}{7}(-2, 6, 3)$, $\mathbf{e}_2 = \frac{1}{7}(6, 3, -2)$ and so $r_{1,1} = \langle \mathbf{e}_1, \mathbf{v}_1\rangle = 7$, $r_{1,2} = \langle \mathbf{e}_1, \mathbf{v}_2\rangle = 7$, $r_{2,2} = \langle \mathbf{e}_2, \mathbf{v}_2\rangle = 7$. So $Q = \frac{1}{7}\begin{bmatrix} -2 & 6 \\ 6 & 3 \\ 3 & -2 \end{bmatrix}$, $R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$.

(d) For $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (1, 1, 1)$, $\mathbf{v}_3 = (1, 3, 0)$ get $\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = \frac{1}{\sqrt{2}}(1, 0, -1)$ and so $r_{1,1} = \langle \mathbf{e}_1, \mathbf{v}_1\rangle = \sqrt{2}$, $r_{1,2} = \langle \mathbf{e}_1, \mathbf{v}_2\rangle = \sqrt{2}$, $r_{1,3} = \langle \mathbf{e}_1, \mathbf{v}_3\rangle = 1/\sqrt{2}$, $r_{2,2} = \langle \mathbf{e}_2, \mathbf{v}_2\rangle = 1$, $r_{2,3} = \langle \mathbf{e}_2, \mathbf{v}_3\rangle = 3$, $r_{3,3} = \langle \mathbf{e}_3, \mathbf{v}_3\rangle = 1/\sqrt{2}$. So $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$, $R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 3 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$.

(e) For $\mathbf{v}_1 = (2, 0, 2, 0)$, $\mathbf{v}_2 = (2, 0, 2, 1)$ get $\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, 0, 1, 0)$, $\mathbf{e}_2 = (0, 0, 0, 1)$ and so $r_{1,1} = \langle \mathbf{e}_1, \mathbf{v}_1\rangle = 2\sqrt{2}$, $r_{1,2} = \langle \mathbf{e}_1, \mathbf{v}_2\rangle = 2\sqrt{2}$, $r_{2,2} = \langle \mathbf{e}_2, \mathbf{v}_2\rangle = 1$. So $Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 0 & 1 \end{bmatrix}$.

(f) For $\mathbf{v}_1 = (2, 0, 2, 1)$, $\mathbf{v}_2 = (2, 0, 2, 0)$ get $\mathbf{e}_1 = \frac{1}{3}(2, 0, 2, 1)$, $\mathbf{e}_2 = \frac{1}{\sqrt{18}}(1, 0, 1, -4)$ and so $r_{1,1} = \langle \mathbf{e}_1, \mathbf{v}_1\rangle = 3$, $r_{1,2} = \langle \mathbf{e}_1, \mathbf{v}_2\rangle = 8/3$, $r_{2,2} = \langle \mathbf{e}_2, \mathbf{v}_2\rangle = 4/\sqrt{18}$. So $Q = \begin{bmatrix} 2/3 & 1/\sqrt{18} \\ 0 & 0 \\ 2/3 & 1/\sqrt{18} \\ 1/3 & -4/\sqrt{18} \end{bmatrix}$, $R = \begin{bmatrix} 3 & 8/3 \\ 0 & 4/\sqrt{18} \end{bmatrix}$.
