1. Identify each statement as either true or false:
   (a) Every matrix has a unique row-echelon form.
   (b) For $n \times n$ matrices A and B, $(A+B)^2 = A^2 + 2AB + B^2$.
   (c) For an invertible $n \times n$ matrix A, $(A^4)^{-1} = (A^{-1})^4$.
   (d) The inverse of the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$.
   (e) For any $n \times n$ matrix A, $\det(A) = -\det(A^T)$.
   (f) If B is obtained from A by adding 4 times the first row of A to the third row of A, then $\det(B) = 4\det(A)$.
   (g) For any 3×3 matrix A, if B is obtained by tripling every entry of A, then $\det(B) = 3\det(A)$.
   (h) If the coefficient matrix of a 6×6 system is invertible, then the system has infinitely many solutions.
   (i) The set $S = \{\mathbf{v}, \mathbf{w}\}$ is linearly independent whenever $\mathbf{v}$ and $\mathbf{w}$ are both nonzero.
   (j) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a spanning set for V, then so is $\{\mathbf{v}_1, \mathbf{v}_3\}$.
   (k) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, then so is $\{\mathbf{v}_1, \mathbf{v}_3\}$.
   (l) If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for V, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
   (m) If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
   (n) If $\dim(V) = 3$, then a set of 2 vectors in V cannot span V.
   (o) If $\dim(V) = 3$, then any spanning set for V can contain at most 3 vectors.
   (p) If $\dim(V) = 3$, then a set of 2 vectors in V cannot be linearly independent.
   (q) If $\dim(V) = 3$, then there is a linearly independent subset of V having exactly 3 elements.
   (r) If $\dim(V) = 3$, then any basis of V must contain exactly 3 vectors.
   (s) If A is an invertible $n \times n$ matrix, then the rows of A are linearly independent.
   (t) Every finite-dimensional vector space has a finite basis.
   (u) A linearly independent set of vectors is a basis of V if and only if it spans V.
   (v) If A is an $m \times n$ matrix, the row space is a subspace of $\mathbb{R}^m$.
   (w) If A is an $m \times n$ matrix, the column space is a subspace of $\mathbb{R}^m$.
   (x) If A is an $m \times n$ matrix, the nullspace is a subspace of $\mathbb{R}^m$.
   (y) If A is an $m \times n$ matrix, the row space of A has the same dimension as the nullspace of A.
   (z) If A is an invertible $n \times n$ matrix, then the row space of A must be $\mathbb{R}^n$.

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2. Solve the following systems of linear equations:

(a) $\left\{ \begin{array}{rcl} x+y-z & = & 5 \\ 3x+y+3z & = & 3 \\ 4x+y+5z & = & 1 \end{array} \right\}$

(b) $\left\{ \begin{array}{rcl} 2x-4y+3z & = & 5 \\ x-2y+z & = & 3 \\ -3x+y+2z & = & 1 \end{array} \right\}$

(c) $\left\{ \begin{array}{rcl} a-b+2c-d & = & 2 \\ 3a+b-c+d & = & 4 \\ -2a-2c & = & 6 \end{array} \right\}$

(d) $\left\{ \begin{array}{rcl} a+b+c+d+e & = & 1 \\ a+2b+3c+4d+5e & = & 6 \end{array} \right\}$

(e) $\left\{ \begin{array}{rcl} a-b-3c+7d & = & -4 \\ 2a-2b+c & = & 6 \\ 3a-3b+c+d & = & 8 \\ -2a+2b+c-4d & = & -2 \end{array} \right\}$

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3. Calculate the determinant of each matrix:

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 & 3 & -1 \\ 0 & 2 & 0 & 0 \\ 2 & -4 & 1 & 3 \\ 0 & 6 & -2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 5 & 4 \\ 3 & 4 & 1 \\ -3 & 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 3 & 1 \\ 2 & 7 & 6 & -2 \\ 2 & -1 & -1 & -3 \\ 4 & 2 & 3 & 4 \end{bmatrix}$

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4. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix}$.

(a) Find $\det(A)$ and $\det(B)$.
(b) Find A^{-1} and B^{-1}.
(c) Find $\det(A^T BAB^T)$ and $\det(5B^2)$.
(d) Find $\mathbf{v}$ if $A\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

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5. If $\mathbf{v} = \langle 3, 1, -1, 3 \rangle$ and $\mathbf{w} = \langle 1, 1, 1, -1 \rangle$, find $\mathbf{v} - 3\mathbf{w}$, $\mathbf{v} \cdot \mathbf{w}$, and $||\mathbf{v}||$.

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6. In each of the following, determine whether the given subset S is a subspace of the given vector space V: if so, verify the subspace criterion, and if not, identify a counterexample.

(a) $V = \mathbb{R}^3$, $S = \{\langle a, b, c \rangle \, : \, a + b + c = 0\}$.
(b) $V = \mathbb{R}^3$, $S = \{\langle x, y, z \rangle \, : \, x^2 + y^2 + z^2 = 1\}$.
(c) $V = \mathbb{R}^5$, $S =$ the vectors $\langle a, b, c, d, e \rangle$ with $e = a + b$ and $b = c = d$.
(d) $V = M_{2\times 2}(\mathbb{R})$, $S = \{A \in V : \det(A) = 0\}$.
(e) $V = M_{2\times 2}(\mathbb{R})$, $S = \{A \in V : A^2 = A\}$.
(f) $V = M_{3\times 3}(\mathbb{R})$, $S = \{A \in V : A^T = A\}$.
(g) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with nonnegative real entries.

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7. For each of the following sets of vectors inside a vector space V, determine (i) if S is linearly independent, (ii) if S spans V, and (iii) if S is a basis of V:

(a) $V = \mathbb{R}^2$, $S = \{(1, 3),\ (2, 4),\ (3, 1)\}$.
(b) $V = \mathbb{R}^3$, $S = \{(1, 1, 0),\ (2, 0, -1)\}$.
(c) $V = \mathbb{R}^3$, $S = \{(1, 1, 0),\ (2, 0, -1),\ (0, 2, 1)\}$.
(d) $V = \mathbb{R}^3$, $S = \{(1, 1, 0),\ (2, 0, -1),\ (0, 2, 1),\ (0, 1, -1)\}$.
(e) $V = M_{2\times 2}(\mathbb{R})$, $S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.
(f) $V = P_3(\mathbb{R})$, $S = \{1 - x,\ 3 - x^2,\ 4 - x^3\}$.
(g) $V = P_2(\mathbb{R})$, $S = \{1 - x - x^2,\ 3 + 3x + 2x^2,\ 4 + x + x^2\}$.

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8. For each matrix, find the reduced row-echelon form, the rank, and also find bases for the row space, column space, and nullspace:

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 6 & 8 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 & -1 & -3 \\ 3 & 6 & 2 & 1 & 5 \\ 4 & 8 & 4 & 2 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 3 & -3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 4 & -2 & 0 & 6 \\ 3 & 6 & -3 & 1 & 9 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 2 \\ 1 & 1 & 4 & 6 \\ 1 & -1 & 8 & 8 \end{bmatrix}$

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9. Find a basis, and the dimension, of each of the given vector spaces:

(a) $V = 3 \times 3$ lower-triangular matrices.
(b) $V = 3 \times 3$ symmetric matrices with diagonal entries 0.
(c) $V =$ the plane $x - 2y + 2z = 0$ in $\mathbb{R}^3$.
(d) $V =$ the span of $\langle 1, 1, 1, 1 \rangle, \langle 1, 1, 2, 2 \rangle, \langle 2, 2, 0, 0 \rangle, \langle 3, 3, 5, 5 \rangle$ in $\mathbb{R}^4$.
