1. (a) False: although the reduced row-echelon form is unique, row-echelon forms are not.
   (b) False: the correct statement is $(A+B)^2 = A^2 + AB + BA + B^2$.
   (c) True: note that $A^4(A^{-1})^4 = I_n$ by repeatedly cancelling, so the inverse of A^4 is $(A^{-1})^4$.
   (d) True: multiplying out gives $\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so they are inverses.
   (e) False: in general $\det(A) = \det(A^T)$, not $-\det(A^T)$.
   (f) False: this row operation doesn't change the determinant, so in fact $\det(B) = \det(A)$ here.
   (g) False: all 3 rows are tripled, so the determinant is scaled by $3^3 = 27$ rather than 3.
   (h) False: if the coefficient matrix is invertible, there is a unique solution.
   (i) False: for example if $\mathbf{w} = 2\mathbf{v}$ then the set can be linearly dependent.
   (j) False: a counterexample is $\mathbf{v}_1 = \langle 1,0,0\rangle$, $\mathbf{v}_2 = \langle 0,1,0\rangle$, $\mathbf{v}_3 = \langle 0,0,1\rangle$.
   (k) True: removing vectors from a linearly independent set preserves independence.
   (l) True: adding vectors to a spanning set still gives a spanning set.
   (m) False: a counterexample is $\mathbf{v}_1 = \langle 1,0\rangle$, $\mathbf{v}_2 = \langle 0,1\rangle$, $\mathbf{v}_3 = \langle 1,1\rangle$.
   (n) True: if $\dim(V) = 3$ then at least 3 vectors are needed to span V.
   (o) False: if $\dim(V) = 3$, any spanning set for V must contain at least 3 vectors (not at most 3).
   (p) False: a counterexample is $\mathbf{v}_1 = \langle 1,0,0\rangle$ and $\mathbf{v}_2 = \langle 0,1,0\rangle$.
   (q) True: if $\dim(V) = 3$ then there is a basis with 3 elements, and bases are linearly independent.
   (r) True: if $\dim(V) = 3$, then every basis for V has exactly 3 vectors.
   (s) True: row operations don't affect independence of rows, and the reduced row-echelon form is the identity.
   (t) True: this is what it means for a vector space to be finite-dimensional.
   (u) True: this is just a rephrasing of the definition of a basis.
   (v) False: the rows have length n, so the row space is a subspace of $\mathbb{R}^n$.
   (w) True: the columns have length m, so the column space is a subspace of $\mathbb{R}^m$.
   (x) False: vectors in the nullspace have the same length as vectors in the rowspace, so the nullspace is a subspace of $\mathbb{R}^n$.
   (y) False: the row space and column space always have the same dimension, not the row space and nullspace.
   (z) True: row operations do not change the row space, and the reduced row-echelon form is the identity matrix (whose rows are a basis for $\mathbb{R}^n$).

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2. In each case, write down the augmented coefficient matrix and then use row operations to solve the system.
   (a) No solution (inconsistent).
   (b) $(x,y,z) = (-2,-3,-1)$.
   (c) $(a,b,c,d) = (3,-11-d,-6,d)$.
   (d) $(a,b,c,d,e) = (-4+c+2d+3e, 5-2c-3d-4e)$.
   (e) $(a,b,c,d) = (2+b-d, b, 2+2d, d)$.

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3. Use row operations / expansion along rows to get (a) 0 (b) 56 (c) 93 (d) 8

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4. (a) $\det(A) = -2$ and $\det(B) = 4$.

(b) $A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -1 & 1 \\ -3 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $B^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ -3 & 6 & 5 \\ 2 & -4 & -2 \end{bmatrix}$.

(c) $\det(A^T BAB^T) = \det(A^T)\det(B)\det(A)\det(B^T) = \det(A)^2\det(B)^2 = 64$, $\det(5B^2) = 5^3\det(B)^2 = 2000$.

(d) Multiplying on the left by A^{-1} gives $\mathbf{v} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 & 1 \\ -3 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

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5. $\mathbf{v} - 3\mathbf{w} = \langle 0, -2, -4, 6\rangle$, $\mathbf{v}\cdot\mathbf{w} = 0$, $||\mathbf{v}|| = \sqrt{20}$.

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6. For each of these, just check the three parts [S1]-[S3] of the subspace criterion. On the exam, you would be expected to justify these responses.

(a) Subspace: all three parts hold. [S1] $\langle 0,0,0\rangle$ is in S, [S2] if $\langle a_1, b_1, c_1\rangle$, $\langle a_2, b_2, c_2\rangle$ in S so $a_1 + b_1 + c_1 = 0 = a_2 + b_2 + c_2$, adding gives $(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = 0$ so $\langle a_1 + a_2, b_1 + b_2, c_1 + c_2\rangle$ in S, [S3] if $\langle a, b, c\rangle$ in S so $a + b + c = 0$ then $ra + rb + rc = 0$ so $r\langle a, b, c\rangle$ in S.

(b) Not a subspace: none of [S1]-[S3] holds.

(c) Subspace: all three parts hold. [S1] $\langle 0,0,0,0,0\rangle$ is in S, [S2] if $\langle a_1, b_1, c_1, d_1, e_1\rangle$, $\langle a_2, b_2, c_2, d_2, e_2\rangle$ in S so $e_1 = a_1 + b_1$, $e_2 = a_2 + b_2$, $b_1 = c_1 = d_1$, $b_2 = c_2 = d_2$, then $e_1 + e_2 = (a_1 + a_2) + (b_1 + b_2)$, $b_1 + b_2 = c_1 + c_2 = d_1 + d_2$ so $\langle a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2, e_1, e_2\rangle$ in S, [S3] if $\langle a, b, c, d, e\rangle$ in S so $e = a + b$ and $b = c = d$ then $re = ra + rb$ and $rb = rc = rd$ so $r\langle a, b, c, d, e\rangle$ in S.

(d) Not a subspace: [S2] does not hold. For example, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ are in S, but $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ is not.

(e) Not a subspace: [S2] and [S3] do not hold. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in S, but $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not.

(f) Subspace: all three parts hold. [S1] the zero matrix is in S, [S2] if A, B in S so $A^T = A$ and $B^T = B$, then $(A + B)^T = A^T + B^T = A + B$ so $A + B$ is in S, [S3] if A in S then $(rA)^T = r(A^T) = rA$ so rA is in S.

(g) Not a subspace: [S3] does not hold. For example, the identity matrix is in S, but its negative is not.

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7. To test linear independence, check whether there are nonzero solutions to $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, and to test spanning, check whether $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ always has a solution for every vector $\mathbf{w}$. On the exam, you would be expected to justify these responses.

(a) Not linearly independent (3 vectors in 2-dimensional space), spans V (matrix with vectors as columns has rank 2), not a basis (not independent).

(b) Linearly independent (2 vectors neither of which is a multiple of the other), does not span V (2 vectors in 3-dimensional space), not a basis (not spanning).

(c) Not linearly independent, does not span V, not a basis (determinant of matrix with vectors as columns is 0, meaning not independent and not spanning).

(d) Not linearly independent (4 vectors in 3-dimensional space), spans V (matrix with vectors as columns has rank 3), not a basis (not independent).

(e) Linearly independent ($a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ forces $a = b = c = d = 0$), spans V ($\begin{bmatrix} p & q \\ r & s \end{bmatrix} = p\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (q-p)\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + (r-q)\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + (s-r)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$), basis.

(f) Linearly independent ($a(1-x)+b(3-x^2)+c(4-x^3)=0$ has only the solution $a=b=c=0$), does not span V (3 vectors in 4-dimensional space), not a basis (not spanning).

(g) Linearly independent ($a(1-x-x^2)+b(3+3x+2x^2)+c(4+x+x^2)=0$ has only the solution $a=b=c=0$), spans V ($p+qx+rx^2=\frac{p+5q-9r}{5}(1-x-x^2)+(q-r)(3+3x+2x^2)+\frac{p-5q+6r}{5}(4+x+x^2)$), basis.

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8. First, row-reduce the matrix. The rank is the number of pivots. For the row space, take the nonzero rows of the row-reduction. For the column space, take the pivot columns in the original matrix. For the nullspace, write down the solutions to the homogeneous system $A\mathbf{x}=\mathbf{0}$.

(a) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, rank 2. Row $\langle 1,0,-1,-2\rangle, \langle 0,1,2,3\rangle$, col $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, null $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, rank 3. Row $\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle$, col $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, null $\emptyset$ (empty basis).

(c) $\begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$, rank 3. Row $\langle 1,2,0,0,4\rangle, \langle 0,0,1,0,-7\rangle, \langle 0,0,0,1,7\rangle$, col $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$,
null $\begin{bmatrix} -4 \\ 0 \\ 7 \\ -7 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, rank 2. Row $\langle 1,0,-3\rangle, \langle 0,1,2\rangle$, col $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, null $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, rank 2. Row $\langle 1,2,-1,0,3\rangle$, $\langle 0,0,0,1,0\rangle$, col $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, null $\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(f) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, rank 3. Row $\langle 1,0,0,1\rangle, \langle 0,1,0,1\rangle, \langle 0,0,1,1\rangle$, col $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}$, null $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

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9. (a) Basis $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, dimension 6.

(b) Basis $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, dimension 3.

(c) Basis $\langle -2,0,1\rangle, \langle 2,1,0\rangle$, dimension 2.

(d) Basis $\langle 1,1,0,0\rangle, \langle 0,0,1,1\rangle$, dimension 2.
