1. To find a formula for M^n we diagonalize M by writing $M = QDQ^{-1}$ and then $M^n = PD^nP^{-1}$.

   (a) Eigenvalues are $\lambda = 1, 2$ with eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   Diagonalizable with $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, so $Q^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$.

   Then $M^n = QD^nQ^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \cdot 2^n - 2 \\ 0 & 2^n \end{bmatrix}$.

   (b) Eigenvalues are $\lambda = 2, 4$ with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   Diagonalizable with $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, so $Q^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

   Then $M^n = QD^nQ^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4^n & 0 \\ 0 & 2^n \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4^n + 2^n & 4^n - 2^n \\ 4^n - 2^n & 4^n + 2^n \end{bmatrix}$.

   (c) Eigenvalues are $\lambda = 3, 8$ with eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

   Diagonalizable with $D = \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$, so $Q^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$.

   Then $M^n = QD^nQ^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 8^n & 0 \\ 0 & 3^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \cdot 8^n + 3 \cdot 3^n & 6 \cdot 8^n - 6 \cdot 3^n \\ 8^n - 3^n & 2 \cdot 8^n + 2 \cdot 3^n \end{bmatrix}$.

   (d) Eigenvalues are $\lambda = 1, 0, -1$ with eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Diagonalizable with $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, so $Q^{-1} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. Then $M^n = QD^nQ^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -2 + 4(-1)^n & 1 - 2(-1)^n & -1 + 3(-1)^n \\ -4 + 4(-1)^n & 2 - 2(-1)^n & -2 + 3(-1)^n \end{bmatrix}$.

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2. The entries $a_{i,j}$ in the associated matrix have $a_{i,i}$ equal to the coefficient of x_i^2 and $a_{i,j}$ equal to half the coefficient of $x_i x_j$ with $i \neq j$. For definiteness, simply find the eigenvalues of the associated matrix and check signs.

   (a) Matrix $\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$, eigenvalues $2 \pm \sqrt{2}$. Eigenvalues all positive, so Q is positive-definite.

   (b) Matrix $\begin{bmatrix} 0 & -2 \\ -2 & -3 \end{bmatrix}$, eigenvalues $-4, 1$. Both a positive and a negative eigenvalue, so Q is indefinite.

   (c) Matrix $\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$, eigenvalues $-3 \pm \sqrt{5}$. Eigenvalues all negative, so Q is negative-definite.

   (d) Matrix $\begin{bmatrix} 4 & -6 \\ -6 & 9 \end{bmatrix}$, eigenvalues $0, 13$. Eigenvalues all nonnegative and one is zero, so Q is positive-semidefinite.

   (e) Matrix $\begin{bmatrix} 7 & 4 & -2 \\ 4 & 7 & -2 \\ -2 & -2 & 4 \end{bmatrix}$, eigenvalues $12, 3, 3$. Eigenvalues all positive, so Q is positive-definite.

   (f) Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, eigenvalues $6, \pm\sqrt{3}$. Both a positive and a negative eigenvalue, so Q is indefinite.

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3. The decomposition $M = U\Sigma V^T$ has Σ the rectangular diagonal matrix with singular values on the diagonal, V have orthonormal columns $\mathbf{v}_i$ given by the corresponding unit eigenvectors of $M^T M$, and U have orthonormal columns given by $\mathbf{w}_i = M\mathbf{v}_i/\sigma_i$ (possibly extended by Gram-Schmidt if there are not enough vectors).

(a) $M^T M = \begin{bmatrix} 13 & 18 \\ 18 & 40 \end{bmatrix}$, eigenvalues $\lambda = 49, 4$ (so $\sigma_1 = 7$, $\sigma_2 = 2$) with corresponding unit eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

(b) $M^T M = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$, eigenvalues $\lambda = 85, 0$ (so $\sigma_1 = \sqrt{85}$) with corresponding unit eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and completing to an orthonormal basis gives $\mathbf{w}_2 = \frac{1}{\sqrt{17}}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$. Thus $U = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

(c) $M^T M = \begin{bmatrix} 8 & 14 & 18 \\ 14 & 29 & 36 \\ 18 & 36 & 45 \end{bmatrix}$, eigenvalues $\lambda = 81, 1, 0$ (so $\sigma_1 = 9$, $\sigma_2 = 1$) with corresponding unit eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{45}}\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 2/\sqrt{45} & -2/\sqrt{5} & -1/3 \\ 4/\sqrt{45} & 1/\sqrt{5} & -2/3 \\ 5/\sqrt{45} & 0 & 2/3 \end{bmatrix}$.

(d) This is the transpose of the matrix in (c) so the SVD just has U and V swapped and Σ transposed: $U = \begin{bmatrix} 2/\sqrt{45} & -2/\sqrt{5} & -1/3 \\ 4/\sqrt{45} & 1/\sqrt{5} & -2/3 \\ 5/\sqrt{45} & 0 & 2/3 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 9 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

(e) $M^T M = \begin{bmatrix} 5 & -4 & -1 \\ -4 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, eigenvalues $\lambda = 9, 3, 0$ (so $\sigma_1 = 3$, $\sigma_2 = \sqrt{3}$) with corresponding unit eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus $U = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$, $V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$.

(f) $M^T M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$, eigenvalues $\lambda = 5, 1, 0, 0$ (so $\sigma_1 = \sqrt{5}$, $\sigma_2 = 1$) with corresponding unit eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$ (note that it's necessary to do Gram-Schmidt to get an orthonormal basis for the 0-eigenspace since it's 2-dimensional). Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{10} & -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{15} \\ 1/\sqrt{10} & 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{15} \\ 2/\sqrt{10} & 0 & 0 & -3/\sqrt{15} \\ 2/\sqrt{10} & 0 & 1/\sqrt{3} & 2/\sqrt{15} \end{bmatrix}$.
