MATH 2331: Linear Algebra -- Final Exam

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Name: Solutions and Grading Key

Directions: This exam is in two parts: multiple choice and fairly short open response questions. Show all work in the space provided.

Part I: Multiple Choice (10 points total: @1 point)

Circle first, and then print in the black space (at the end of each question), the CAPITAL LETTERS corresponding to the correct answers. (No need to show work.)

#1. The area of the parallelogram spanned by the vectors $\boldsymbol{v} = (5,4)$ and $\boldsymbol{w} = (3,2)$ is
(A) -22 (B) 22 (C) 23 (D) 8 (E) 2 E

#2. Let P be a parallelogram, having area 6 square units, spanned by two linearly independent vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by $T(x,y) = (2x+3y, 4x+5y)$, then the area of the image parallelogram $T(P)$ is
(A) 192 (B) 132 (C) -132 (D) 12 (E) 6 D

#3. Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & -2 & 3 \\ 1 & 0 & 4 \end{bmatrix}$. What is $tr(A)$, the trace of A?
(A) 3 (B) 5 (C) 6 (D) 7 (E) 24 B

#4. Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$. What is $det(A)$, the determinant of A?
(A) 0 (B) -5 (C) 5 (D) -24 (E) 24 D

#5. Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$. What are the eigenvalues of A?
(A) $3, -2, 4$ (B) $-3, 2, -4$ (C) $1, 6, 7$ (D) $3, 4, 7$ (E) $-2, -3, 5$ A

#6. Let W be the subspace of $\mathbb{R}^3$ spanned by the two vectors $(1,-1,1)$ and $(3,-2,0)$. Which one of the following vectors is a basis for $W^{\perp}$, the orthogonal complement of W?
(A) $(1,1,0)$ (B) $(-1,0,1)$ (C) $(1,2,1)$ (D) $(2,3,0)$ (E) $(2,3,1)$ E

#7. The eigenvalues of $A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ are
(A) $-1, 3$ (B) $2, 2$ (C) $-7, \frac{1}{7}$ (D) $2 \pm 4i$ (E) $1 \pm 2\sqrt{2}$ E

#8. The eigenvalues of $A = \begin{bmatrix} 4 & 5 \\ -5 & -4 \end{bmatrix}$ are
(A) $4, -4$ (B) $-3, 5$ (C) $\pm 3i$ (D) $1 \pm 4i$ (E) $-1 \pm \sqrt{5}$ C

#9. Which of the following is <u>not</u> necessarily a valid factorization of the given matrix M?
(A) if M is any square matrix, then $M = QR$, where Q and R are both orthogonal matrices
(B) if M has linearly independent columns, then $M = QR$ where Q has orthonormal columns and R is an invertible upper triangular matrix
(C) if M is a real symmetric matrix, then $M = QDQ^T$ for some orthogonal matrix Q and diagonal matrix D
(D) if M is any matrix of rank r, then $M = U\Sigma V^T$ for some orthogonal matrices U, V and scalar matrix Σ of rank r A

#10. For the Singular Value Decomposition of an arbitrary matrix M, which one of these statements is false?
(A) MM^T and M^TM are symmetric (B) MM^T and M^TM have the same size (C) a real symmetric matrix has real eigenvalues (D) two othogonal matrices U and V are needed (E) a scalar matrix of eigenvalue square roots is needed B

Part II: Open Responses (90 points total)

(14 pts) #1. Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and let $\boldsymbol{b} = (5, -9, 4)$.

8 (a) First solve the non-homogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} = (x_1, x_2, x_3)$, expressing your final answer in parametric form, and then answer parts (b,c,d,e,f) below.

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ -3 & -8 & 7 & -9 \\ 2 & 5 & -3 & 4 \end{array}\right] \xrightarrow[-2R_1+R_3 \;\; 1]{1 \;\; 3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 0 & 1 & -5 & 6 \\ 0 & -1 & 5 & -6 \end{array}\right] \xrightarrow[1]{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

x_3 is free.

1 $R_2: x_2 - 5x_3 = 6 \Rightarrow x_2 = 5x_3 + 6$

2 $R_1: x_1 + 3(5x_3+6) - 4x_3 = 5 \Rightarrow x_1 = -11x_3 - 13.$

2 $\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11x_3 - 13 \\ 5x_3 + 6 \\ x_3 \end{bmatrix} = \underbrace{t\begin{bmatrix} -11 \\ 5 \\ 1 \end{bmatrix}}_{1} + \underbrace{\begin{bmatrix} -13 \\ 6 \\ 0 \end{bmatrix}}_{1}$

1 (b) What is $rank(A)$? 2

1 (c) What is $nullity(A)$? 1

1 (d) State a basis for $ker(A) = NS(A)$: $\{(-11, 5, 1)\}$

2 (e) State a basis for $im(A) = CS(A)$: $\{(1,-3,2), (3,-8,5)\}$

1 (f) What is $dim(im\,A)$? 2

(5 pts) #2. Given the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (10x + 3y, 6x + 2y)$, find the vector or coordinate formula for the inverse linear transformation $T^{-1}(x, y)$.

$M_T = \begin{bmatrix} 10 & 3 \\ 6 & 2 \end{bmatrix} \Rightarrow M_T^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -3 & 5 \end{bmatrix}$ 3

$\Rightarrow T^{-1}(x,y) = \begin{bmatrix} 1 & -\frac{3}{2} \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - \frac{3}{2}y \\ -3x + 5y \end{bmatrix}$ 2

or $(x - \frac{3}{2}y, -3x + 5y)$

(8 pts) #3. Given the scaled rotation matrix $R = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and the scaled reflection matrix $F = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

4 (a) Find the matrix of the composition $T = R \circ F$.

$$M_T = RF = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \underline{\begin{bmatrix} 5 & 5 \\ 5 & -5 \end{bmatrix}}$$

4 (b) Find the matrix of the composition $S = F \circ R$.

$$M_S = FR = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \underline{\begin{bmatrix} -1 & 7 \\ 7 & 1 \end{bmatrix}}$$

(8 pts) #4. Define the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix}$, where $A = \begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix}$ is the matrix of T in the standard basis $\boldsymbol{e_1} = (1, 0)$, $\boldsymbol{e_2} = (0, 1)$ for $\mathbb{R}^2$. If the basis for $\mathbb{R}^2$ is changed to $\boldsymbol{v_1} = (2, 1)$, $\boldsymbol{v_2} = (5, 3)$, what is the matrix representing T in this new basis?

1 Let $S = [v_1 \; v_2] = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. If $B = M_T$ in the new basis, then $B = \underset{2}{S^{-1}}\underbrace{AS}_{1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -5 & 9 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \underline{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}.$$

4

(10 pts) #5. Let $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -6 \\ 2 & 8 & -3 \end{bmatrix}$.

8 (a) Calculate A^{-1} (showing work steps!) by the Gauss-Jordan method, checking your result, and then answer parts (b) and (c) below.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 4 & -6 & 0 & 1 & 0 \\ 2 & 8 & -13 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 2 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\xrightarrow[4R_3+R_1]{2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 8 & -4 \\ 0 & 1 & 0 & -1 & 5 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -7 & 2 \\ 0 & 1 & 0 & -1 & 5 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

Check: $\begin{bmatrix} 4 & -7 & 2 \\ -1 & 5 & -2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -6 \\ 2 & 8 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$$\therefore A^{-1} = \begin{bmatrix} 4 & -7 & 2 \\ -1 & 5 & -2 \\ 0 & 2 & -1 \end{bmatrix}$$

1 (b) Based on your work above, what is $rank(A)$? 3

1 (c) Based on your work above, what is $det(A)$? -1

(there was one negation: $-R_3$)

(6 pts) #6. Prove that the vectors $v_1 = (1, -3, 4)$, $v_2 = (-2, 7, 6)$, $v_3 = (7, -23, 0)$ are linearly dependent, expressing one of them as a linear combination of the others.

Find c_1, c_2, c_3 such that $c_1 V_1 + c_2 V_2 + c_3 V_3 = \overline{0}$, so

$$\begin{bmatrix} 1 & -2 & 7 \\ -3 & 7 & -23 \\ 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ -3 & 7 & -23 & 0 \\ 4 & 6 & 0 & 0 \end{array}\right] \xrightarrow[-4R_1+R_3]{3R_1+R_2}$$

3

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 14 & -28 & 0 \end{array}\right] \xrightarrow{-14R_2+R_3} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \end{array}.$$

$\therefore c_3$ is free; R_2: $c_2 - 2c_3 = 0$ or $\underline{c_2 = 2c_3}$

R_1: $c_1 - 2(2c_3) + 7c_3 = 0 \Rightarrow \underline{c_1 = -3c_3}$.

Take $c_3 = 1$, so $c_2 = 2$, $c_1 = -3$.

3

Claim: $-3V_1 + 2V_2 + V_3 = \overline{0}$, so $\underline{V_3 = 3V_1 - 2V_2}$.

Check: $3V_1 - 2V_2 = 3\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ -23 \\ 0 \end{bmatrix} = V_3$ ✓

(6 pts) #7. Given that the three 3-dimensional vectors $v_1 = (-1, 3, -4)$, $v_2 = (3, -8, 10)$, $v_3 = (2, -9, 7)$ are linearly independent (hence form a basis for $\mathbb{R}^3$), express the vector $w = (1, -14, 5)$ as a linear combination of these three vectors v_1, v_2, v_3, checking that your answer really works.

Find c_1, c_2, c_3 such that $c_1 V_1 + c_2 V_2 + c_3 V_3 = W$, or

$$\begin{bmatrix} -1 & 3 & 2 \\ 3 & -8 & -9 \\ -4 & 10 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \\ 5 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 3 & -8 & -9 & -14 \\ -4 & 10 & 7 & 5 \end{array}\right]$$

3

$$\xrightarrow[-4R_1+R_3]{3R_1+R_2} \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 1 & -3 & -11 \\ 0 & -2 & -1 & 1 \end{array}\right] \xrightarrow{2R_2+R_3} \begin{array}{c} \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 1 & -3 & -11 \\ 0 & 0 & -7 & -21 \end{array}\right] \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \end{array} \Rightarrow -7c_3 = -21,$$

so $\underline{c_3 = 3}$

Back-substitute: from R_2, $c_2 - 3(3) = -11 \Rightarrow \underline{c_2 = -2}$;

from R_1: $-c_1 + 3(-2) + 2(3) = 1 \Rightarrow \underline{c_1 = -1}$

3

Claim: $\underline{W = -V_1 - 2V_2 + 3V_3}$

Check: $-V_1 - 2V_2 + 3V_3 = -\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -8 \\ 10 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -9 \\ 7 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -14 \\ 5 \end{bmatrix} = W$ ✓

(11 pts) #8. Let W be the subspace of $\mathbb{R}^3$ spanned by the two linearly independent vectors $v_1 = (-1, 2, 2)$ and $v_2 = (3, -3, 0)$.

5 (a) Use the Gram-Schmidt orthogonalization process to find an orthonormal basis for W.

1 Let $u_1 = \frac{1}{3}(-1,2,2)$.

3 Let $v_2^{\perp} = v_2 - (u_1 \cdot v_2)u_1 = (3,-3,0) - \frac{(-9)}{3} \cdot \frac{(-1,2,2)}{3}$

$= (3,-3,0) + (-1,2,2) = (2,-1,2)$.

Check: $u_1 \cdot v_2^{\perp} = 0$, so let $u_2 = \frac{1}{3}(2,-1,2)$. 1

Then $\{u_1, u_2\}$ is an ON-basis for W:

$u_1 = \frac{1}{3}(-1,2,2)$, $u_2 = \frac{1}{3}(2,-1,2)$.

5 (b) Use part (a) to find the matrix M of the orthogonal projection $P : \mathbb{R}^3 \to W$.

2 Let $Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}$, an orthogonal matrix,

3 Then $M_P = QQ^T = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}$, a symmetric matrix.

1 (c) Given that $im(P) = W$, what is $rank(M)$? Since $\dim W = 2$, $\dim im(P) = 2$.

$\therefore \dim CS(M_P) = 2$. $\therefore$ rank $M = 2$.

($\therefore P$ is not invertible.)

(6 pts) #9. Showing work, evaluate the following determinant by using the properties of elementary row or column operations instead of the Laplace expansion by minors: $\begin{vmatrix} 2 & 1 & -4 & 0 \\ 7 & 3 & -13 & 8 \\ -5 & -2 & 11 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix}$.

4 $\left\{ \begin{array}{l} \overset{C_1 \leftrightarrow C_2}{=} -\begin{vmatrix} 1 & 2 & -4 & 0 \\ 3 & 7 & -13 & 8 \\ -2 & -5 & 11 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix} \\ \underset{2R_1+R_3}{\overset{-3R_1+R_2}{=}} -\begin{vmatrix} 1 & 2 & -4 & 0 \\ 0 & 1 & -1 & 8 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix} \overset{R_2+R_3}{=} -\begin{vmatrix} 1 & 2 & -4 & 0 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 6 \end{vmatrix} \end{array} \right.$

2 $\{ \; = -(1)(1)(2)(6) = \underline{-12}$

(6 pts) #10. Given that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 4 & -4 & 10 \\ 0 & 0 & 2 \end{bmatrix}$, find an associated eigenvector corresponding to $\lambda = 2$.

1 $A - \lambda I_3 = \begin{bmatrix} 3-\lambda & -1 & 1 \\ 4 & -4-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{bmatrix} \Longrightarrow$

1 $A - 2I = \begin{bmatrix} 1 & -1 & 1 \\ 4 & -6 & 10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[1]{-4R_1+R_2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} r & s & t \end{array} \end{array}$ } 3

3 $\left\{ \begin{array}{l} \therefore t \text{ is free; } R_2: -2s+6t=0 \Rightarrow \underline{s=3t}; \\ R_1: r-3t+t=0 \Rightarrow \underline{r=2t}. \\ \therefore V_{\lambda=2} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \end{array} \right.$

The simplest eigenvector for $\lambda = 2$ is $\underline{(2,3,1)}$, a basis.
(In fact, $E_2 = \text{span}\{(2,3,1)\}$.)

(10 pts) #11. Given the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, find an orthonormal basis under which A is similar to a diagonal matrix D, also indicating the entries of D.

By the Spectral Theorem A can be diagonalized over $\mathbb{R}$.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

2 $$= (1-\lambda)(1-\lambda) - 4 = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \underline{\lambda = 3, -1}.$$

$\underline{\lambda = 3}$: $A - 3I | \bar{0} = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ (s, t)

2 $\therefore$ t is free and $s - t = 0$ or $s = t$.

$\therefore V_{\lambda=3} = \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = t\, v_1.$

$\underline{\lambda = -1}$: $A + I | \bar{0} = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ (s, t)

2 $\therefore$ t is free and $s + t = 0$ or $s = -t$.

$\therefore V_{\lambda=-1} = \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = t\, v_2.$

(Note that $v_1 \cdot v_2 = 0$.)

2 Let $u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1)}{\sqrt{2}}$, $u_2 = \frac{v_2}{\|v_2\|} = \frac{(-1,1)}{\sqrt{2}}$.

Let $S = [u_1 \ u_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then S is an orthogonal matrix.

2 Under the change of basis $\{u_1, u_2\}$,

$$A \sim S^{-1}AS = S^T A S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D,$$

with the eigenvalues of A along the main diagonal of D.