MATH 2331: Linear Algebra -- Final Exam

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Name: ______________________________

Directions: This exam is in two parts: multiple choice and fairly short open response questions. Show all work in the space provided.

Part I: Multiple Choice (10 points total: @1 point)

Circle first, and then print in the black space (at the end of each question), the CAPITAL LETTERS corresponding to the correct answers. (No need to show work.)

#1. The area of the parallelogram spanned by the vectors $\boldsymbol{v} = (5, 4)$ and $\boldsymbol{w} = (3, 2)$ is
(A) -22 (B) 22 (C) 23 (D) 8 (E) 2 ____

#2. Let P be a parallelogram, having area 6 square units, spanned by two linearly independent vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by $T(x, y) = (2x + 3y, 4x + 5y)$, then the area of the image parallelogram $T(P)$ is
(A) 192 (B) 132 (C) -132 (D) 12 (E) 6 ____

#3. Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & -2 & 3 \\ 1 & 0 & 4 \end{bmatrix}$. What is $tr(A)$, the trace of A?
(A) 3 (B) 5 (C) 6 (D) 7 (E) 24 ____

#4. Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$. What is $det(A)$, the determinant of A?
(A) 0 (B) -5 (C) 5 (D) -24 (E) 24 ____

#5. Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$. What are the eigenvalues of A?
(A) $3, -2, 4$ (B) $-3, 2, -4$ (C) $1, 6, 7$ (D) $3, 4, 7$ (E) $-2, -3, 5$ ____

#6. Let W be the subspace of $\mathbb{R}^3$ spanned by the two vectors $(1, -1, 1)$ and $(3, -2, 0)$. Which one of the following vectors is a basis for $W^{\perp}$, the orthogonal complement of W?
(A) $(1, 1, 0)$ (B) $(-1, 0, 1)$ (C) $(1, 2, 1)$ (D) $(2, 3, 0)$ (E) $(2, 3, 1)$ ____

#7. The eigenvalues of $A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ are
(A) $-1, 3$ (B) $2, 2$ (C) $-7, \frac{1}{7}$ (D) $2 \pm 4i$ (E) $1 \pm 2\sqrt{2}$ ____

#8. The eigenvalues of $A = \begin{bmatrix} 4 & 5 \\ -5 & -4 \end{bmatrix}$ are
(A) $4, -4$ (B) $-3, 5$ (C) $\pm 3i$ (D) $1 \pm 4i$ (E) $-1 \pm \sqrt{5}$ ____

#9. Which of the following is <u>not</u> necessarily a valid factorization of the given matrix M?
(A) if M is any square matrix, then $M = QR$, where Q and R are both orthogonal matrices
(B) if M has linearly independent columns, then $M = QR$ where Q has orthonormal columns and R is an invertible upper triangular matrix
(C) if M is a real symmetric matrix, then $M = QDQ^T$ for some orthogonal matrix Q and diagonal matrix D
(D) if M is any matrix of rank r, then $M = U\Sigma V^T$ for some orthogonal matrices U, V and scalar matrix Σ of rank r ____

#10. For the Singular Value Decomposition of an arbitrary matrix M, which one of these statements is false?
(A) MM^T and M^TM are symmetric (B) MM^T and M^TM have the same size (C) a real symmetric matrix has real eigenvalues (D) two othogonal matrices U and V are needed (E) a scalar matrix of eigenvalue square roots is needed ____

Part II: Open Responses (90 points total)

(14 pts) #1. Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and let $\boldsymbol{b} = (5, -9, 4)$.

(a) First solve the non-homogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} = (x_1, x_2, x_3)$, expressing your final answer in parametric form, and then answer parts (b,c,d,e,f) below.

(b) What is $rank(A)$? __________

(c) What is $nullity(A)$? __________

(d) State a basis for $ker(A) = NS(A)$: ____________________

(e) State a basis for $im(A) = CS(A)$: ____________________

(f) What is $dim(im\ A)$? __________

(5 pts) #2. Given the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (10x + 3y, 6x + 2y)$, find the vector or coordinate formula for the inverse linear transformation $T^{-1}(x, y)$.

(8 pts) #3. Given the scaled rotation matrix $R = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and the scaled reflection matrix $F = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

(a) Find the matrix of the composition $T = R \circ F$.

(b) Find the matrix of the composition $S = F \circ R$.

(8 pts) #4. Define the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix}$, where $A = \begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix}$ is the matrix of T in the standard basis $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$ for $\mathbb{R}^2$. If the basis for $\mathbb{R}^2$ is changed to $\boldsymbol{v}_1 = (2, 1)$, $\boldsymbol{v}_2 = (5, 3)$, what is the matrix representing T in this new basis?

(10 pts) #5. Let $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -6 \\ 2 & 8 & -3 \end{bmatrix}$.

(a) Calculate A^{-1} (showing work steps!) by the Gauss-Jordan method, checking your result, and then answer parts (b) and (c) below.

(b) Based on your work above, what is $rank(A)$?

(c) Based on your work above, what is $det(A)$?

(6 pts) #6. Prove that the vectors $v_1 = (1, -3, 4)$, $v_2 = (-2, 7, 6)$, $v_3 = (7, -23, 0)$ are linearly dependent, expressing one of them as a linear combination of the others.

(6 pts) #7. Given that the three 3-dimensional vectors $v_1 = (-1, 3, -4)$, $v_2 = (3, -8, 10)$, $v_3 = (2, -9, 7)$ are linearly independent (hence form a basis for $\mathbb{R}^3$), express the vector $w = (1, -14, 5)$ as a linear combination of these three vectors v_1, v_2, v_3, checking that your answer really works.

(11 pts) #8. Let W be the subspace of $\mathbb{R}^3$ spanned by the two linearly independent vectors $v_1 = (-1, 2, 2)$ and $v_2 = (3, -3, 0)$.

(a) Use the Gram-Schmidt orthogonalization process to find an orthonormal basis for W.

(b) Use part (a) to find the matrix M of the orthogonal projection $P : \mathbb{R}^3 \to W$.

(c) Given that $im(P) = W$, what is $rank(M)$?

(6 pts) #9. Showing work, evaluate the following determinant by using the properties of elementary row or column operations instead of the Laplace expansion by minors: $\begin{vmatrix} 2 & 1 & -4 & 0 \\ 7 & 3 & -13 & 8 \\ -5 & -2 & 11 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix}$.

(6 pts) #10. Given that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 4 & -4 & 10 \\ 0 & 0 & 2 \end{bmatrix}$, find an associated eigenvector corresponding to $\lambda = 2$.

(10 pts) #11. Given the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, find an orthonormal basis under which A is similar to a diagonal matrix D, also indicating the entries of D.