NORTHEASTERN UNIVERSITY
Department of Mathematics

MATH 2331 (Linear Algebra) **Final Exam — Spring 2017**

Do not write in these boxes:

Problem	1	2	3	4	5	6	7	8	9	10	11	Total
Score												
Max Pts	11	6	6	12	9	5	8	11	5	12	15	100

Name: Solutions **Instructor:** ________

Instructions:

- Write your name and your instructor's name in the blanks above.
- **SHOW YOUR WORK.** If there is not enough room to show your work, use the back of the preceding page.
- **Sufficient work must be shown to justify answers. No calculator or any other reference is permitted.**

SOLUTIONS

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1. (a) (10 points) Consider the system:

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & - & x_2 & - & 5x_3 & & & & -3x_5 & + & 2x_6 & = & 8 \\
2x_1 & - & x_2 & - & 2x_3 & + & x_4 & & -11x_5 & + & 4x_6 & = & 17 \\
 & & & & & & 2x_4 & & -6x_5 & & -x_6 & = & 2 \\
-x_1 & + & x_2 & + & 5x_3 & + & x_4 & & & & -3x_6 & = & -8
\end{array}$$

Use Gauss-Jordan elimination to compute the $rref$ of the augmented matrix of the system. Show all necessary steps of the computation. Then, use the $rref$ to find all common solutions of the system. Indicate which unknowns, if any, act as free variables. Using the free variables as parameters, write down all common solutions of the system.

$$\left[\begin{array}{cccccc|c} 1 & -1 & -5 & 0 & -3 & 2 & 8 \\ 2 & -1 & -2 & 1 & -11 & 4 & 17 \\ 0 & 0 & 0 & 2 & -6 & -1 & 2 \\ -1 & 1 & 5 & 1 & 0 & -3 & -8 \end{array}\right] \to \begin{array}{c} \\ \\ R_4 \\ R_3 \end{array}\left[\begin{array}{cccccc|c} 1 & -1 & -5 & 0 & -3 & 2 & 8 \\ 2 & -1 & -2 & 1 & -11 & 4 & 17 \\ -1 & 1 & 5 & 1 & 0 & -3 & -8 \\ 0 & 0 & 0 & 2 & -6 & -1 & 2 \end{array}\right] \to$$

$$\begin{array}{c} \\ R_2-2R_1 \\ R_3+R_1 \\ \\ \end{array}\left[\begin{array}{cccccc|c} 1 & -1 & -5 & 0 & -3 & 2 & 8 \\ 0 & 1 & 8 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 2 & -6 & -1 & 2 \end{array}\right] \to \begin{array}{c} \\ \\ \\ R_4-2R_3 \end{array}\left[\begin{array}{cccccc|c} 1 & -1 & -5 & 0 & -3 & 2 & 8 \\ 0 & 1 & 8 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\begin{array}{c} R_1-2R_4 \\ \\ R_3+R_4 \\ \\ \end{array}\left[\begin{array}{cccccc|c} 1 & -1 & -5 & 0 & -3 & 0 & 4 \\ 0 & 1 & 8 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to \begin{array}{c} \\ R_2-R_3 \\ \\ \\ \end{array}\left[\begin{array}{cccccc|c} 1 & -1 & -5 & 0 & -3 & 0 & 4 \\ 0 & 1 & 8 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\boxed{\begin{array}{c} R_1+R_2 \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccccc|c} 1 & 0 & 3 & 0 & -5 & 0 & 3 \\ 0 & 1 & 8 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \\ \begin{array}{cccccc} & & \uparrow & & \uparrow & \\ & & r & & s & \end{array}}$$

This matrix is rref. Set $x_3 = r$, $x_5 = s$ free variables $\Rightarrow$

$$\boxed{\begin{aligned} x_1 &= 3 - 3r + 5s \\ x_2 &= -1 - 8r + 2s \\ x_3 &= r \\ x_4 &= 2 + 3s \\ x_5 &= s \\ x_6 &= 2 \end{aligned}} \text{ or } \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} -3 \\ -8 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}}$$

(b) (1 point) Suppose that

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix}$$

are two solutions of the system of equations in part (a).

Is the following statement TRUE or FALSE? **Choose one or the other as your answer.**

"$\vec{v} + \vec{w}$ is a solution of the system of equations in part(a)."

$\boxed{\text{False}}$

Set A = coefficient matrix of the system. Then $Av = \begin{bmatrix} 8 \\ 17 \\ 2 \\ -8 \end{bmatrix}$ and $Aw = \begin{bmatrix} 8 \\ 17 \\ 2 \\ -8 \end{bmatrix}$.

But then, $A(v+w) = Av + Aw = \begin{bmatrix} 8 \\ 17 \\ 2 \\ -8 \end{bmatrix} + \begin{bmatrix} 8 \\ 17 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 16 \\ 34 \\ 4 \\ -16 \end{bmatrix}$, which is not $\begin{bmatrix} 8 \\ 17 \\ 2 \\ -8 \end{bmatrix}$,

so $v + w$ is not a solution of the system.

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2. (6 points) Given the matrix $M = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Use Gauss-Jordan elimination to compute the inverse of M.

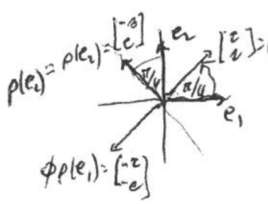
$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - 2R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 - R_3 \\ R_2 + 2R_3 \\ -R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \quad \therefore \; M^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check: $MM^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

3. In this problem, you may use: $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \sin(\frac{\pi}{4})$.

(a) (3 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which begins with first rotation, ρ, counter-clockwise about the origin by angle $\theta = \frac{\pi}{4}$ followed by reflection ϕ across the line $y = -x$. Find the matrix of T with respect to the standard basis.



$$e_1 \overset{\rho}{\longmapsto} \begin{bmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix} \overset{\phi}{\longmapsto} \begin{bmatrix} -\cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = T(e_1)$$

$$e_2 \overset{\rho}{\longmapsto} \begin{bmatrix} -\sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} \end{bmatrix} \overset{\phi}{\longmapsto} \begin{bmatrix} -\sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = T(e_2)$$

$$\therefore [T] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) (3 points) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which begins with first rotation, ρ, counterclockwise about the origin by angle $\theta = \frac{\pi}{4}$ followed by orthogonal projection π onto the line $y = -x$. Find the matrix of S in the standard basis.

$$e_1 \overset{\rho}{\longmapsto} \begin{bmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix} \overset{\pi}{\longmapsto} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = S(e_1)$$

$$e_2 \longmapsto \begin{bmatrix} -\sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} \end{bmatrix} \overset{\pi}{\longmapsto} \begin{bmatrix} -\sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = S(e_2)$$

$$\therefore [S] = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

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4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 3 & 5 & 1 \\ 1 & 2 & 2 & 3 & 0 \end{bmatrix}$

(a) (2 points) The matrix $B = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & x & y & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the $rref$ of A. Find x and y.

$$B = rref(A) \Longrightarrow \boxed{x = 0 \text{ and } y = 1} \Longrightarrow B = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (3 points) Find a basis for $Im(A)$. Give reasons for your answer.

Leading variables in column 1 and 3 of $B = rref(A)$.

Hence Basis for $Im(A) = \{v_1, v_3\} = \boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}}$

(where v_j = jth column of A.)

(c) (5 points) Find a basis for $ker(A)$. Show your work.

$$\text{From } B = rref(A) = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \quad & \uparrow r & \quad & \uparrow s & \uparrow t \end{array} \end{array} \Longrightarrow \text{Ker } A \text{ is s.t. } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r - s - 2t \\ r \\ -s + t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \left[\right.$$

$$\text{so basis for } Ker(A) = \boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

(d) (2 points) Find a basis for $(Im(A^T))^\perp$. Give reasons for your answer.

$Im(A^T)$ = span columns of A^T = span rows (transposed) of A = $Row(A)$

$\therefore (Im(A^T))^\perp = (Row(A))^\perp = Ker(A)$

$$\therefore \text{basis for } (Im(A^T))^\perp = \boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

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5. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$. The columns of A are $\{\vec{v_1},\ \vec{v_2},\ \vec{v_3}\}$.

(a) (6 points) Use the Gram-Schmidt process to find a matrix Q with orthonormal columns $\{\vec{q_1},\ \vec{q_2},\ \vec{q_3}\}$ such that $Im(Q) = Im(A)$.

$w_1 = v_1,\ w_2 = v_2.$

$$w_3 = v_3 - \left(\frac{v_3 \cdot w_1}{w_1 \cdot w_1}\right) w_1 - \left(\frac{v_3 \cdot w_2}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 4 \end{bmatrix} - \left(\frac{9}{9}\right)\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{9}{9}\right)\begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

Set $u_j = \frac{w_j}{\|w_j\|}$. Then $Q = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 2/2 & 1/3 & 0 \\ 1/3 & -2/3 & 2/3 \\ 0 & 2/3 & 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

(b) (3 points) Using the matrix Q found in part (a), find a matrix R so that $A = QR$. Clearly label the matrix R.

$$R = Q^T M = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ -1 & 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -2 & 1 \\ 0 & 2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & 0 & 9 \\ 0 & 9 & 9 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

6. Suppose that V is a subspace of $\mathbf{R}^4$ with orthonormal basis $\{\vec{q_1},\ \vec{q_2}\}$ where

$$\vec{q_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{q_2} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1, \end{bmatrix}$$

(a) (3 points) Find the matrix P of the orthogonal projection $\mathbf{R}^4$ onto V.

Observe $q_1 \perp q_2$ and $\|q_1\| = 1 = \|q_2\|$. Form $Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$. Then $P = QQ^T$

$$P = QQ^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) (2 point) Calculate the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ onto V.

$$P v = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 8 \\ 2 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

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7. Consider the matrix

$$M = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -7 & -5 \\ -5 & 10 & 8 \end{bmatrix}.$$

(a) (3 points) Find the eigenvalues of M.

$$\det(M-\lambda I) = \det\begin{bmatrix} (3-\lambda) & 0 & 0 \\ 5 & (-7-\lambda) & -5 \\ -5 & 10 & (8-\lambda) \end{bmatrix} = (3-\lambda)\det\begin{bmatrix} (-7-\lambda) & -5 \\ 10 & (8-\lambda) \end{bmatrix}$$

$$= (3-\lambda)\left[(-7-\lambda)(8-\lambda)+50\right] = (3-\lambda)(-56-\lambda+\lambda^2+50) = (3-\lambda)(-6-\lambda+\lambda^2)$$

$$= (3-\lambda)(3-\lambda)(-2-\lambda) \Rightarrow \boxed{\lambda = 3, 3, -2} \text{ eigenvalues}$$

(b) (3 points) Find a basis of eigenvectors for each eigenspace of M

$\lambda=3$: $(M-3I)v=\vec{0} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 5 & -10 & -5 \\ -5 & 10 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a\begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} + b\begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix} + c\begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow a=1, b=0, c=1$ or $a=2, b=1, c=0$ for example $\Rightarrow$ basis for E_3 is $\boxed{\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}}$

$\lambda=-2$: $(M+2I)v=\vec{0} \Rightarrow \begin{bmatrix} 5 & 0 & 0 \\ 5 & -5 & -5 \\ -5 & 10 & 10 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a\begin{bmatrix} 5 \\ 5 \\ -5 \end{bmatrix} + b\begin{bmatrix} 0 \\ -5 \\ 10 \end{bmatrix} + c\begin{bmatrix} 0 \\ -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow a=0, b=1, c=-1$ for example $\Rightarrow$ basis for E_{-2} is $\boxed{\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}}$

(c) (2 points) Explain why we know that M is invertible.

As $\lambda=0$ is not an eigenvalue, there do not exist nonzero vector v s.t. $Mv = 0v = \vec{0}$. Hence $\ker(M) = \{\vec{0}\} \Rightarrow M$ invertible.

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8. Given

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix},$$

(a) (8 points) Find the least–squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$,

$$A^T A x^* = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$ Hence, must solve

$$\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 2 & 2 & 8 \end{array}\right] \xrightarrow{\substack{\frac{1}{2}R_2 \\ R_1}} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & 2 & 10 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow{\substack{R_1 + R_2 \\ -R_2}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \boxed{x^* = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

(b) (3 points) $Im(A)$ is a plane in $\mathbb{R}^3$ that passes through the origin. Find the distance from the point (2, 3, 5) to $Im(A)$.

Let π = orthogonal projection onto $Im(A)$. Then $Ax^* = \pi(b)$.

With $b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\pi(b) = Ax^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$.

Distance from b to $Im(A) = \|b - \pi(b)\| = \left\| \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{0+1+1} = \boxed{\sqrt{2}}$

9. (5 points) Evaluate $det(K)$ (in terms of a) if $K = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 0 & 5 & 3 \\ 1 & 2a & 2 & 2 \\ -1 & 0 & a & 4 \end{bmatrix}$, ~~in terms of a, b and c~~.

$$det(K) = (-1)^{3+2}(2a)\, det\begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 3 \\ -1 & a & 4 \end{bmatrix} = -2a\left(2\, det\begin{bmatrix} 5 & 3 \\ a & 4 \end{bmatrix} - det\begin{bmatrix} 3 & 3 \\ -1 & 4 \end{bmatrix}\right)$$

$$= -2a\big(2(20-3a) - (12+3)\big) = -2a(40 - 6a - 15) = -2a(25 - 6a)$$

$$= \boxed{-50a + 12a^2}$$

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10. Let G be a 3×3 matrix whose eigenspaces are: $\mathcal{E}_1 = span\left\{\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$, and $\mathcal{E}_2 = span\left\{\begin{bmatrix}-3\\0\\6\end{bmatrix}, \begin{bmatrix}4\\-5\\2\end{bmatrix}\right\}$.

(a) (6 points) Find a factorization of G of the form $G = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is a diagonal matrix. Clearly identify the matrices Q, Q^T, and Λ, showing all the entries of each matrix. Do not multiply together $G = Q\Lambda Q^T$ or explicitly find G.

Normalizing these vectors, $\boxed{Q = \begin{bmatrix} 2/3 & -3/\sqrt{45} & 4/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 1/3 & 6/\sqrt{45} & 2/\sqrt{45} \end{bmatrix}}$ and $\boxed{\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$

$$\boxed{Q^T = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -3/\sqrt{45} & 0 & 6/\sqrt{48} \\ 4/\sqrt{45} & -5/\sqrt{45} & 2/\sqrt{45} \end{bmatrix}}$$

(b) (3 points) Referring to part (a), what is the factorization for G^{-1} that corresponds to the factorization in part(a).

$$G^{-1} = (Q\Lambda Q^T)^{-1} = (Q^T)^{-1}\Lambda^{-1}Q^{-1} = \boxed{Q\Lambda^{-1}Q^T}$$

$$= \boxed{\underbrace{\begin{bmatrix} 2/3 & -3/\sqrt{45} & 4/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 1/3 & 6/\sqrt{45} & 2/\sqrt{45} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{\Lambda^{-1}} \underbrace{\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -3/\sqrt{45} & 0 & 6/\sqrt{65} \\ 4/\sqrt{45} & -5/\sqrt{45} & 2/\sqrt{45} \end{bmatrix}}_{Q^T}}$$

(c) (3 points) If n is a positive integer, set $G^{-n} = (G^{-1})^n$. Use the factorization of part (b) to explicitly compute the matrix $H = \lim_{n\to\infty} G^{-n}$. H should be written as a single matrix, not a product of matrices.

$$H = \lim_{n\to\infty}(Q\Lambda^{-1}Q^T) = Q\left(\lim_{n\to\infty}\Lambda^{-1}\right)Q^T = Q\left(\lim_{n\to\infty}\begin{bmatrix} (1)^n & 0 & 0 \\ 0 & (\frac{1}{2})^n & 0 \\ 0 & 0 & (\frac{1}{3})^n \end{bmatrix}\right)Q^T$$

$$= Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}Q^T = \begin{bmatrix} 2/3 & -3/\sqrt{45} & 4/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 1/3 & 6/\sqrt{45} & 2/\sqrt{45} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -3/\sqrt{45} & 0 & 6/\sqrt{45} \\ 4/\sqrt{45} & -5/\sqrt{45} & 2/\sqrt{45} \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 0 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & 1/3 \\ \frac{-3}{\sqrt{45}} & 0 & \frac{6}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} & \frac{-5}{\sqrt{45}} & \frac{2}{\sqrt{45}} \end{bmatrix} = \boxed{\begin{bmatrix} 4/9 & 4/9 & 2/9 \\ 4/9 & 4/9 & 2/9 \\ 2/9 & 2/9 & 1/9 \end{bmatrix}} = \boxed{\frac{1}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}}$$

SOLUTIONS

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11. Consider $C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$. Let $N = C^T C$.

(a) (2 points) Find N.

$$N = C^T C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}}$$

(b) (3 points) Show that the following vectors are eigenvectors for N and determine the corresponding eigenvalues.

$$\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{b_3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$N b_1 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{\lambda = 4} \text{ eigenvalue with eigenvector } b_1$$

$$N b_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \boxed{\lambda = 2} \text{ eigenvalue with eigenvector } b_2$$

$$N b_3 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \boxed{\lambda = 0} \text{ eigenvalue with eigenvector } b_3$$

(c) (3 points) Find an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ for $\mathbb{R}^3$ consisting of eigenvectors for N.

Set $v_j = \frac{b_j}{\|b_j\|}$. An orthonormal basis is $\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$

(d) (1 point) Find the singular values of C.

$\sigma_j = \sqrt{\lambda_j}$ so $\boxed{\sigma = 2, \sqrt{2}, 0}$

SOLUTIONS

(Continuation of problem 10 from previous page)

(e) (3 points) Determine the vectors $C\vec{v_1}$, $C\vec{v_2}$, $C\vec{v_3}$.

$$Cv_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \boxed{\begin{bmatrix} 2/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix}}$$

$$Cv_2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$Cv_3 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

(f) (3 points) Using your work in (a)-(e), determine the matrices V, Σ, U of the singular value factorization of A (you do not need to multiply U, Σ and V^T). Label each matrix clearly.

$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$ where $u_j = \frac{1}{\sigma_j} C v_j$, so $u_1 = \frac{1}{2} \begin{bmatrix} 2/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

and $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Hence $U = \boxed{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$

$$\boxed{\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}}$$

$$V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}}$$