NORTHEASTERN UNIVERSITY
Department of Mathematics

MATH 2331 (Linear Algebra) **Final Exam — Spring 2017**

Do not write in these boxes:

Problem	1	2	3	4	5	6	7	8	9	10	11	Total
Score												
Max Pts	11	6	6	12	9	5	8	11	5	12	15	100

Name: ______________________ **Instructor:** ______________________

Instructions:

- Write your name and your instructor's name in the blanks above.
- **SHOW YOUR WORK.** If there is not enough room to show your work, use the back of the preceding page.
- **Sufficient work must be shown to justify answers. No calculator or any other reference is permitted.**

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1. (a) (10 points) Consider the system:

$$\begin{array}{rcrcrcrcrcrcr} x_1 & - & x_2 & - & 5x_3 & & & & -3x_5 & + & 2x_6 & = & 8 \\ 2x_1 & - & x_2 & - & 2x_3 & + & x_4 & & -11x_5 & + & 4x_6 & = & 17 \\ & & & & & & 2x_4 & & -6x_5 & - & x_6 & = & 2 \\ -x_1 & + & x_2 & + & 5x_3 & + & x_4 & & & - & 3x_6 & = & -8 \end{array}$$

Use Gauss-Jordan elimination to compute the $rref$ of the augmented matrix of the system. Show all necessary steps of the computation. Then, use the $rref$ to find all common solutions of the system. Indicate which unknowns, if any, act as free variables. Using the free variables as parameters, write down all common solutions of the system.

(b) (1 point) Suppose that

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix}$$

are two solutions of the system of equations in part (a).

Is the following statement TRUE or FALSE? **Choose one or the other as your answer.**

"$\vec{v} + \vec{w}$ is a solution of the system of equations in part(a)."

2. (6 points) Given the matrix $M = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Use Gauss-Jordan elimination to compute the inverse of M. Label the inverse of M clearly.

3. In this problem, you may use: $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \sin(\frac{\pi}{4})$.

   (a) (3 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which begins with first rotation, ρ, counterclockwise about the origin by angle $\theta = \frac{\pi}{4}$ followed by reflection ϕ across the line $y = -x$. Find the matrix of T with respect to the standard basis.

   (b) (3 points) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which begins with first rotation, ρ, counterclockwise about the origin by angle $\theta = \frac{\pi}{4}$ followed by orthogonal projection π onto the line $y = -x$. Find the matrix of S in the standard basis.

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 3 & 5 & 1 \\ 1 & 2 & 2 & 3 & 0 \end{bmatrix}$

(a) (2 points) The matrix $B = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & x & y & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the $rref$ of A. Find x and y.

(b) (3 points) Find a basis for $Im(A)$. Give reasons for your answer.

(c) (5 points) Find a basis for $ker(A)$. Show your work.

(d) (2 points) Find a basis for $\left(Im(A^T)\right)^{\perp}$. Give reasons for your answer.

5. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$. The columns of A are $\{\vec{v_1},\ \vec{v_2},\ \vec{v_3}\}$.

   (a) (6 points) Use the Gram-Schmidt process to find a matrix Q with orthonormal columns $\{\vec{q_1},\ \vec{q_2},\ \vec{q_3}\}$ such that $Im(Q) = Im(A)$.

   (b) (3 points) Using the matrix Q found in part (a), find a matrix R so that $A = QR$. Clearly label the matrix R.

6. Suppose that V is a subspace of $\mathbf{R}^4$ with orthonormal basis $\{\vec{q_1},\ \vec{q_2}\}$ where

$$\vec{q_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{q_2} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1, \end{bmatrix}$$

   (a) (3 points) Find the matrix P of the orthogonal projection $\mathbf{R}^4$ onto V.

   (b) (2 point) Calculate the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ onto V.

7. Consider the matrix

$$M = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -7 & -5 \\ -5 & 10 & 8 \end{bmatrix} .$$

(a) (3 points) Find the eigenvalues of M.

(b) (3 points) Find a basis of eigenvectors for each eigenspace of M.

(c) (2 points) Explain why we know that M is invertible.

8. Given

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix},$$

(a) (8 points) Find the least–squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$,

(b) (3 points) $Im(A)$ is a plane in $\mathbb{R}^3$ that passes through the origin. Find the distance from the point $(2,\ 3,\ 5)$ to $Im(A)$.

9. (5 points) Evaluate $det(K)$ in terms of a if $K = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 0 & 5 & 3 \\ 1 & 2a & 2 & 2 \\ -1 & 0 & a & 4 \end{bmatrix}$.

10. Let G be a 3×3 matrix whose eigenspaces are: $\mathcal{E}_1 = span\left\{\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$, and $\mathcal{E}_2 = span\left\{\begin{bmatrix}-3\\0\\6\end{bmatrix}, \begin{bmatrix}4\\-5\\2\end{bmatrix}\right\}$.

(a) (6 points) Find a factorization of G of the form $G = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is a diagonal matrix. Clearly identify the matrices Q, Q^T, and Λ, showing all the entries of each matrix. Do not multiply together $G = Q\Lambda Q^T$ or explicitly find G.

(b) (3 points) Referring to part (a), what is the factorization for G^{-1} that corresponds to the factorization in part(a).

(c) (3 points) If n is a positive integer, set $G^{-n} = (G^{-1})^n$. Use the factorization of part (b) to explicitly compute the matrix $H = \lim_{n\to\infty} G^{-n}$. H should be written as a single matrix, not a product of matrices.

11. Consider $C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$. Let $N = C^T C$.

(a) (2 points) Find N.

(b) (3 points) Show that the following vectors are eigenvectors for N and determine the corresponding eigenvalues.

$$\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \vec{b_3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(c) (3 points) Find an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ for $\mathbb{R}^3$ consisting of eigenvectors for N.

(d) (1 point) Find the singular values of C.

(Continuation of problem 11 from previous page)

(e) (3 points) Determine the vectors $C\vec{v_1},\ C\vec{v_2},\ C\vec{v_3}$.

(f) (3 points) Using your work in (a)-(e), determine the matrices V, Σ, U of the singular value factorization of A (you do not need to multiply U, Σ and V^T). Label each matrix clearly.