Northeastern University

Math 2331-Linear Algebra-Fall 2020

Final Exam.

Instructor:

Student Name: ____________________ __________**/100**

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown. Only a final result from calculator will receive zero point.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. You are allowed to look at notes or textbook. However, you are **not** allowed to asked help using any online platform.

4. You have 120 minutes for the exam and 15 minutes to scan your solutions, merge into **one .pdf**, and upload. This is plenty of time to use a scanner or scanning app and clearly scan every page.

5. **Late submission will result in a penalty on your final exam grade.** If you have any technical difficulty with the upload or scan, contact me immediately. Do not wait until the end of exam to contact me about a technical difficulty.

1. (10 points) (1) Solve the linear system $\begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 = 3 \\ 2x_3 - 8x_4 = 2 \\ 2x_1 + 12x_2 + 2x_3 - 2x_4 = 4 \end{cases}$ by elementary row operations and write the parametric vector form for all solutions.

(2) Let A be the coefficient matrix of the above linear system, that is $A = \begin{bmatrix} 1 & 6 & 2 & -5 \\ 0 & 0 & 2 & -8 \\ 2 & 12 & 2 & -2 \end{bmatrix}$. Answer the following questions with the help of your above calculation.

(i) A basis for the kernel $\ker(A)$ is ______________________. (ii) A basis for the image $\operatorname{im}(A)$ is ______________________

(iii) A basis for the vector space $(\operatorname{im}(A^T))^{\perp}$ is ____________________________

2. (8 points) Use Elementary Row Operations to find the **inverse** of matrix $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. (Write down all your work)

3. (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation, which begins with first **reflection** about the y-axis, followed by **orthogonal projection** onto the line $y = 3x$. Find the matrix of the transformation T.

4. Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ be a subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$.

(1).(2 point) Is $\{\vec{v}_1, \vec{v}_2\}$ an orthonormal basis for V? Reason.

(2). (5 points) Find the decomposition of $\vec{y} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ as $\vec{y} = \vec{y}_1 + \vec{y}_2$ such that $\vec{y}_1 \in V$ and $\vec{y}_2 \in V^{\perp}$.

(3) (3 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by the orthogonal projection onto V. Find the matrix of the linear transformation T.

5. Suppose $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ form a basis for the vector space W.

(1). (5 points) Using the *Gram-Schmidt process* to $\mathscr{B}$, find an **orthogonal** basis for the vector space W.

(2). (3 points) Normalize the result in (2), find an **orthonormal** basis for the vector space W.

(3). (3 points) Find the QR-factorization of $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

6. Given $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

(1). (4 points) Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$.

(2) (2 points) The image $\text{im}(A)$ is a plane in $\mathbb{R}^3$ passing the origin. Find the distance from the vector $\vec{b}$ (or the point $(1, 2, 3)$) to the plane $\text{im}(A)$.

7. (6 points) Given the transition matrix $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$.

(1) Calculate an eigenvector of A corresponding to eigenvalue 1.

(2) Find $\lim_{t\to\infty} A^t$.

8. Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 8 \\ 1 & 5 & 7 & 9 \\ 0 & 0 & 0 & -2 \\ 2 & 5 & 8 & 10 \end{bmatrix}$. Answer the following questions.

(1). (6 points) Compute the **determinant** of A. Write down all steps you are using.

(2). (4 points) Find the following determinant $\det\left((4A)^{-1}(2A^T)^2\right)$ using the result in (1).

9. (10 points) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

(1) Find a basis for the eigenspace E_λ with eigenvalue $\lambda_1 = 3$.

(2) What are the geometric multiplicities of $\lambda_1 = 3$ and $\lambda_2 = 5$?

(3) Is A diagonalizable? Explain your reason.

(4) Is A orthogonally diagonalizable? Explain your reason.

10. (8 points) Consider matrix $A = \begin{bmatrix} -1 & 5 & 7 \\ 3 & -3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$.

(1) Find all **eigenvalues** for the matrix A. (Write all details of calculation.)

(2) What are the eigenvalues of $A^2 - 3A$?

11. (5 points) The symmetric matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ has eigenvectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ with corresponding eigenvalues $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 0$. Answer the following questions.

(1) Orthogonally diagonalize the matrix A. That is, find an **orthogonal** matrix U and a **diagonal** matrix D such that $A = UDU^{-1} = UDU^T$.

$$U = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \text{ and } D = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}.$$

(2). Let $\vec{a} = \begin{bmatrix} \sqrt{5} \\ \sqrt{3} \\ -\sqrt{3} \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} \sqrt{5} \\ \sqrt{3} \\ 0 \end{bmatrix}$. Answer the following questions.

(i) $||U\vec{a}|| =$ _______________

(ii) $U\vec{a} \cdot U\vec{b} =$ _______________

12. (10 points) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

(1) Compute $B = A^T A$.

12. Continued from page 9.

(2) Verify that the vectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ are eigenvectors of $B = A^T A$. What are their corresponding eigenvalues?

(3) What are the singular values of A?

(4) Calculate the singular value decomposition $A = U\Sigma V^T$. (Clearly write each matrix U, Σ and V.)