1. (10 points) Consider the system:

$$\begin{array}{rcrcrcrcrcrcl} & & & & & & -x_4 & + & 2x_5 & +x_6 & = & 2 \\ x_1 & + & 2x_2 & & & & -x_4 & + & x_5 & & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & & +x_4 & - & x_5 & & = & 2 \end{array}$$

Use Gauss-Jordan elimination to compute the *rref* of the augmented matrix of the system. Show all necessary steps of the computation. Then, use the *rref* to find all solutions of the system. Indicate which unknowns, if any, act as free variables. Using the free variables as parameters, write down all solutions of the system.

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & -1 & 2 & 1 & 2 \\ 1 & 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & -1 & 0 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccccc|c} 1 & 2 & 2 & 1 & -1 & 0 & 2 \\ 1 & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 & 2 \end{array}\right] \xrightarrow{-R_1+R_2}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 2 & 1 & -1 & 0 & 2 \\ 0 & 0 & -2 & -2 & 2 & 0 & -2 \\ 0 & 0 & 0 & -1 & 2 & 1 & 2 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{R_1+R_2} \left[\begin{array}{cccccc|c} 1 & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 1 & 2 \end{array}\right] \xrightarrow{(-1)R_3}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -1 & -2 \end{array}\right] \xrightarrow[R_2+(-1)R_3]{R_1+R_3} \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -2 & -1 & -2 \end{array}\right]$$

rref

x_2, x_5, x_6 are free variables

$x_2 = r$, $x_5 = s$, $x_6 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2r+s+t-2 \\ r \\ -s-t+3 \\ 2s+t-2 \\ s \\ t \end{bmatrix}$$

2. (6 points) Given the matrix $M = \begin{bmatrix} -1 & 0 & 1 \\ 5 & 1 & -6 \\ -7 & -2 & 10 \end{bmatrix}$.

Use Gauss-Jordan elimination to compute the inverse of M. Label the inverse of M clearly.

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 5 & 1 & -6 & 0 & 1 & 0 \\ -7 & -2 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 5 & 1 & -6 & 0 & 1 & 0 \\ -7 & -2 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+7R_1]{R_2-5R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 5 & 1 & 0 \\ 0 & -2 & 3 & -7 & 0 & 1 \end{array}\right] \xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right] \xrightarrow[R_2+R_3]{R_1+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 8 & 3 & 1 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right] \quad \therefore\ M^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 8 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

3. (6 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which begins with first reflection, ϕ, across the line $y = x$, followed by orthogonal projection π onto the line $y = -x$. Find the matrix of T with respect to the standard basis.

$$M_\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_\pi = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$M_T = M_\pi M_\phi = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$. The columns of A are $\vec{v_1}, \vec{v_2}, \dots, \vec{v_5}$.

The matrix

$B = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the *RREF* of A.

(a) (3 points) Find a basis for $Im(A)$. Give reasons for your answer.

Basis : $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}$

$\vec{v_1} \quad \vec{v_3} \quad \vec{v_5}$

Correspond to the columns of rref(A) Containing the leading 1's.

(b) (2 points) Express $\vec{v_4}$ as a linear combination of $\vec{v_2}$ and $\vec{v_3}$.

$$\vec{v_4} = \frac{5}{2}\vec{v_2} - \vec{v_3}$$

(c) (5 points) Find a basis for $ker(A)$. Show your work.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_4 are free variables

$x_2 = s, \quad x_4 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-5t \\ s \\ t \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Basis : $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

5. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 9 \\ 1 & -5 \\ 1 & 3 \end{bmatrix}$. The columns of A are $\{\vec{v_1},\ \vec{v_2}\}$.

(a) (6 points) Use the Gram-Schmidt process to find a matrix Q with orthonormal columns $\{\vec{q_1},\ \vec{q_2}\}$ such that $Im(Q) = Im(A)$.

$$\vec{q_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v_2}^{\perp} = \vec{v_2} - (\vec{q_1} \cdot \vec{v_2})\vec{q_1}$$

$$= \vec{v_2} - 4\vec{q_1}$$

$$= \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}$$

$$\vec{q_2} = \frac{\vec{v_2}^{\perp}}{\|\vec{v_2}^{\perp}\|} = \frac{1}{10}\begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{10} \\ \frac{1}{2} & \frac{7}{10} \\ \frac{1}{2} & -\frac{7}{10} \\ \frac{1}{2} & \frac{1}{10} \end{bmatrix}$$

(b) (3 points) Using the matrix Q found in part (a), find a matrix R so that $A = QR$. Clearly label the matrix R.

$$A = QR = \begin{bmatrix} | & | \\ \vec{q_1} & \vec{q_2} \\ | & | \end{bmatrix}\begin{bmatrix} \|\vec{v_1}\| & \vec{q_1} \cdot \vec{v_2} \\ 0 & \|\vec{v_2}^{\perp}\| \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} 2 & 4 \\ 0 & 10 \end{bmatrix}$$

(c) (2 points) Find the matrix P of the orthogonal projection of $\mathbf{R}^4$ onto $Im(A)$.

$$P = QQ^T$$

$$= \frac{1}{100}\begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$

6. Consider the matrix

$$G = \begin{bmatrix} \frac{2}{6} & 0 & \frac{2}{6} \\ 0 & \frac{4}{6} & \frac{2}{6} \\ \frac{2}{6} & \frac{2}{6} & \frac{3}{6} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

(a) (3 points) Show that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ are eigenvectors of G. For each eigenvector, find the corresponding eigenvalue.

$$G\vec{v}_1 = \frac{1}{6}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \lambda_1 = 1$$

$$G\vec{v}_2 = \frac{1}{6}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \qquad \lambda_2 = \frac{1}{2}$$

$$G\vec{v}_3 = \frac{1}{6}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \qquad \lambda_3 = 0$$

(b) (6 points) Find a factorization of G of the form $G = Q\Lambda Q^T$ with Q an orthogonal matrix and Λ a diagonal matrix. Show all the entries of Q and Λ. Do not multiply together the matrices Q, Λ and Q^T.

$$G = Q\Lambda Q^T \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

(c) (3 points) Use the factorization in part (b) to compute $H = \lim_{n\to\infty} G^n$, explicitly listing its entries.

$$H = \lim_{n\to\infty} G^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{3}\right)^2 \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1^n & 0 & 0 \\ 0 & (\frac{1}{2})^n & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \left(\frac{1}{3}\right)^2 \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

7. Given

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix},$$

(a) (8 points) Find the least–squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$,

$$A^T A \vec{x} = A^T \vec{b}$$

$$\Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 6 & 6 \\ 6 & 14 & 18 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \end{array}\right]$$

$$\vec{x}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

(b) (2 points) Calculate the orthogonal projection of the vector $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$ onto $Im(A)$.

$$\text{proj}_{Im(A)} \vec{b} = A\vec{x}^*$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(c) (3 points) Fit a linear function of t of the form $y = C + Dt$ to the data points (1, 0), (2, 0), (3, 6) using least squares. Explain your reasoning briefly.

the system $\begin{cases} C + D = 0 \\ C + 2D = 0 \\ C + 3D = 6 \end{cases}$ is equivalent to $A\vec{x} = \vec{b}$ with $\vec{x} = \begin{bmatrix} C \\ D \end{bmatrix}$

$$\therefore \begin{bmatrix} C \\ D \end{bmatrix} = \vec{x}^* = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$y = -4 + 3t$$

8. Consider the quadratic form $q = q(x_1, x_2, x_3) = 4(x_1)^2 + 3(x_2)^2 + 4(x_3)^2 + 12x_1x_3$.

(a) (3 points) Find the matrix A of q.

$$A = \begin{bmatrix} 4 & 0 & 6 \\ 0 & 3 & 0 \\ 6 & 0 & 4 \end{bmatrix}$$

(b) (6 points) Determine the definiteness of q (or equvalently, A). Explain your reasoning.

(b) . Find the eigenvalues of A by solving the characteristic equation

$$0 = \begin{vmatrix} 4-\lambda & 0 & 6 \\ 0 & 3-\lambda & 0 \\ 6 & 0 & 4-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 4-\lambda & 6 \\ 6 & 4-\lambda \end{vmatrix} = (3-\lambda)\ [(4-\lambda)^2 - 36]$$

$$= (3-\lambda)(\lambda^2 - 8\lambda + 16 - 36) = (3-\lambda)(\lambda^2 - 8\lambda - 20)$$

$$= (3-\lambda)(\lambda-10)(\lambda+2) \Rightarrow \lambda = 10, 3, -2$$

Since some eigenvalues and some are negative, A is indefinite

9. (a) (5 points) Evaluate $det(3K)$ in terms of a if $K = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 5a & 5 & 5 \\ 2 & 0 & 4 & 3 \\ -1 & 0 & 2a & 3 \end{bmatrix}$.

$$\det K = 5a \det \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 3 \\ -1 & 2a & 3 \end{bmatrix}$$

$$= 5a \left(\det \begin{bmatrix} 4 & 3 \\ 2a & 3 \end{bmatrix} - \det \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix} \right)$$

$$= 5a(12 - 6a - 9) = 5a(3-6a) = 15a - 30a^2$$

$$\det(3K) = 3^4 \det K = 81 \cdot 5a(3-6a) = 405a(3-6a)$$

(b) (2 points) Let A and B be two matrices with $det(A) = -2$, and $det(B) = -3$.
Compute $det(A^3 \cdot B^2 \cdot A^{-1} \cdot B^T)$.

$$\det(A^3 \cdot B^2 \cdot A^{-1} \cdot B^T) = \det(A^3)\det(B^2)\det(A^{-1})\det(B^T)$$

$$= (-2)^3(-3)^2(-\tfrac{1}{2})(-3)$$

$$= -27 \cdot 4 = -108$$

10. Consider $C = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. Let $N = C^T C$.

(a) (2 points) Find N.

$$N = \begin{bmatrix} 2 & 0 \\ 0 & -1 \\ -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(b) (4 points) Show that the following vectors are eigenvectors for $N = C^T C$ and determine the corresponding eigenvalues.

$$\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \vec{b_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{b_4} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$N\vec{b_1} = \begin{bmatrix} 8 \\ 0 \\ -8 \\ 0 \end{bmatrix} = 8\vec{b_1} \qquad \lambda_1 = 8$$

$$N\vec{b_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = 2\vec{b_2} \qquad \lambda_2 = 2$$

$$N\vec{b_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{b_3} \qquad \lambda_3 = 0$$

$$N\vec{b_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{b_4} \qquad \lambda_4 = 0$$

(c) (3 points) Find an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ for $\mathbb{R}^4$ consisting of eigenvectors for N.

$$\vec{v_1} = \frac{1}{\sqrt{2}}\vec{b_1} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \vec{v_2} = \frac{1}{\sqrt{2}}\vec{b_2} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \quad \vec{v_3} = \frac{1}{\sqrt{2}}\vec{b_3} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{v_4} = \frac{1}{\sqrt{2}}\vec{b_4} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

(d) (2 points) Find the singular values of C.

$$\sigma_1 = \sqrt{8} = 2\sqrt{2}$$

$$\sigma_2 = \sqrt{2}$$

$$\sigma_3 = \sigma_4 = 0$$

(Continuation of problem 10 from previous page)

(e) (2 points) Determine the vectors $C\vec{v_1}$, $C\vec{v_2}$, $C\vec{v_3}$, $C\vec{v_4}$.

$$C\vec{v_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

$$C\vec{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$$

$$C\vec{v_3} = C\vec{v_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(f) (3 points) Using your work in (a)-(e), determine the matrices V, Σ, U of the singular value factorization of A (you do not need to multiply U, Σ and V^T). Label each matrix clearly.

$$A = U \Sigma V^T$$

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

$$\frac{1}{\sigma_1} C\vec{v_1} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{1}{\sigma_2} C\vec{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$