NORTHEASTERN UNIVERSITY
Department of Mathematics

MATH 2331 (Linear Algebra) **Final Exam — Fall 2017**

Do not write in these boxes:

Problem	1	2	3	4	5	6	7	8	9	10	Total
Score											
Max Pts	10	6	6	10	11	12	13	9	7	16	100

Name: ______________________ **Instructor:** ______________________

Instructions:

- **Write your name and your instructor's name in the blanks above.**
- **Make sure that your exam has 10 pages.**
- **SHOW YOUR WORK. If there is not enough room to show your work, use the back of the preceding page.**
- **Sufficient work must be shown to justify answers. No calculator or any other reference is permitted.**

1. (10 points) Consider the system:

$$\begin{array}{ccccccccccccc} & & & & & & -x_4 & + & 2x_5 & +x_6 & = & 2 \\ x_1 & + & 2x_2 & & & & -x_4 & + & x_5 & & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & & +x_4 & - & x_5 & & = & 2 \end{array}$$

Use Gauss-Jordan elimination to compute the $rref$ **of the augmented matrix of the system. Show all necessary steps of the computation. Then, use the** $rref$ **to find all solutions of the system. Indicate which unknowns, if any, act as free variables. Using the free variables as parameters, write down all solutions of the system.**

2. (6 points) Given the matrix $M = \begin{bmatrix} -1 & 0 & 1 \\ 5 & 1 & -6 \\ -7 & -2 & 10 \end{bmatrix}$.

Use Gauss-Jordan elimination to compute the inverse of M**. Label the inverse of** M **clearly.**

3. (6 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ **is the linear transformation which begins with first reflection,** ϕ**, across the line** $y = x$**, followed by orthogonal projection** π **onto the line** $y = -x$**. Find the matrix of** T **with respect to the standard basis.**

4. Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$. **The columns of** A **are** $\vec{v_1}$, $\vec{v_2}$, $\ldots$, $\vec{v_5}$.

The matrix

$\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **is the** $RREF$ **of** A.

(a) (3 points) Find a basis for $Im(A)$. **Give reasons for your answer.**

(b) (2 points) Express $\vec{v_4}$ **as a linear combination of** $\vec{v_2}$ **and** $\vec{v_3}$.

(c) (5 points) Find a basis for $ker(A)$. **Show your work.**

5. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 9 \\ 1 & -5 \\ 1 & 3 \end{bmatrix}$**. The columns of** A **are** $\{\vec{v_1},\ \vec{v_2}\}$**.**

(a) **(6 points) Use the Gram-Schmidt process to find a matrix** Q **with orthonormal columns** $\{\vec{q_1},\ \vec{q_2}\}$ **such that** $Im(Q) = Im(A)$**.**

(b) **(3 points) Using the matrix** Q **found in part (a), find a matrix** R **so that** $A = QR$**. Clearly label the matrix** R**.**

(c) **(2 points) Find the matrix** P **of the orthogonal projection of** $\mathbf{R}^4$ **onto** $Im(A)$**.**

6. Consider the matrix

$$\boldsymbol{G} = \begin{bmatrix} \frac{2}{6} & 0 & \frac{2}{6} \\ 0 & \frac{4}{6} & \frac{2}{6} \\ \frac{2}{6} & \frac{2}{6} & \frac{3}{6} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

(a) **(3 points) Show that the vectors** $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, **and** $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ **are eigenvectors of** G**. For each eigenvector, find the corresponding eigenvalue.**

(b) **(6 points) Find a factorization of** G **of the form** $G = Q\Lambda Q^T$ **with** Q **an orthogonal matrix and** Λ **a diagonal matrix. Show all the entries of** Q **and** Λ**. Do not multiply together the matrices** Q, Λ **and** Q^T**.**

(c) **(3 points) Use the factorization in part (b) to compute** $H = \lim_{n\to\infty} G^n$**, explicitly listing its entries.**

7. Given

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix},$$

(a) **(8 points) Find the least–squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$,**

(b) **(2 points) Calculate the orthogonal projection of the vector $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$ onto $Im(A)$.**

(c) **(3 points) Fit a linear function of t of the form $y = C + Dt$ to the data points $(1,\ 0)$, $(2,\ 0)$, $(3,\ 6)$ using least squares. Explain your reasoning briefly.**

8. Consider the quadratic form $\boldsymbol{q} = q(x_1, x_2, x_3) = 4(x_1)^2 + 3(x_2)^2 + 4(x_3)^2 + 12x_1x_3$.

(a) (3 points) Find the matrix A of q.

(b) (6 points) Determine the definiteness of q (or equvalently, A). Explain your reasoning.

9. (a) (5 points) Evaluate $det(3K)$ in terms of a if $\boldsymbol{K} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 5a & 5 & 5 \\ 2 & 0 & 4 & 3 \\ -1 & 0 & 2a & 3 \end{bmatrix}$.

(b) (2 points) Let A and B be two matrices with $det(A) = -2$, and $det(B) = -3$. Compute $det(A^3 \cdot B^2 \cdot A^{-1} \cdot B^T)$.

10. Consider $C = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$**. Let** $N = C^T C$**.**

(a) (2 points) Find N**.**

(b) (4 points) Show that the following vectors are eigenvectors for $N = C^T C$ **and determine the corresponding eigenvalues.**

$$\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \ \vec{b_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \vec{b_4} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \ .$$

(c) (3 points) Find an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \ \vec{v_4}\}$ **for** $\mathbb{R}^4$ **consisting of eigenvectors for** N**.**

(d) (2 points) Find the singular values of C**.**

(Continuation of problem 10 from previous page)

(e) **(2 points) Determine the vectors** $C\vec{v_1},\ C\vec{v_2},\ C\vec{v_3},\ C\vec{v_4}$**.**

(f) **(3 points) Using your work in (a)-(e), determine the matrices** V, Σ, U **of the singular value factorization of** C **(you do not need to multiply** U**,** Σ **and** V^T**). Label each matrix clearly.**