NORTHEASTERN UNIVERSITY
Department of Mathematics

MATH 2331 (Linear Algebra) **Final Exam — Fall 2016**

Do not write in these boxes:

Problem	1	2	3	4	5	6	7	8	9	10	11	Total
Score												
Max Pts	10	7	6	7	12	12	5	5	14	8	14	100

Name: Solutions **Instructor:** ____________________

Instructions:

- Write your name and your instructor's name in the blanks above.
- **SHOW YOUR WORK.** If there is not enough room to show your work, use the back of the preceding page.
- **Sufficient work must be shown to justify answers. No calculator or any other reference is permitted.**

1. (10 points) Consider the system:

$$\begin{array}{rcrcrcrcrcrcr}
2x_1 & + & 4x_2 & + & x_3 & + & x_4 & - & x_5 & + & x_6 & = & 5 \\
x_1 & + & 2x_2 & + & x_3 & + & x_4 & - & x_5 & + & x_6 & = & -3 \\
 & & & & & & x_4 & - & x_5 & & & = & 2 \\
 & & & & & - & x_4 & + & x_5 & + & x_6 & = & 1
\end{array}$$

Find all solutions of the system using Gauss-Jordan elimination on the augmented matrix of the system. Be sure to indicate which variables, if any, are free. Show all steps of your computations.

$$\left[\begin{array}{cccccc|c} 2 & 4 & 1 & 1 & -1 & 1 & 5 \\ 1 & 2 & 1 & 1 & -1 & 1 & -3 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 1 & -1 & 1 & -3 \\ 2 & 4 & 1 & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{array}\right]$$

$$\to \begin{array}{c} \\ R_2-2R_1 \\ \\ R_4+R_3 \end{array} \left[\begin{array}{ccccccc} 1 & 2 & 1 & 1 & -1 & 1 & -3 \\ 0 & 0 & -1 & -1 & 1 & -1 & 11 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2+R_4]{R_1-R_4} \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 1 & -1 & 0 & -6 \\ 0 & 0 & -1 & -1 & 1 & 0 & 14 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\to \begin{array}{c} R_1-R_3 \\ R_2+R_3 \end{array} \left[\begin{array}{ccccccc} 1 & 2 & 1 & 0 & 0 & 0 & -8 \\ 0 & 0 & -1 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[-R_2]{R_1+R_2} \begin{array}{c} \left[\begin{array}{ccccccc} 1 & 2 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \\ \begin{array}{ccccccc} & r & & & s & & \end{array} \end{array}$$

$x_2 = r$ and $x_5 = s$ are free variables

$$\begin{aligned} x_1 &= 8-2r \\ x_3 &= -16 \\ x_4 &= 2+s \\ x_6 &= 3 \end{aligned}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 8-2r \\ r \\ -16 \\ 2+s \\ s \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -16 \\ 2 \\ 0 \\ 3 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

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2. (7 points) Given the matrix $M = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$. Use Gauss-Jordan elimination to compute the inverse of M.

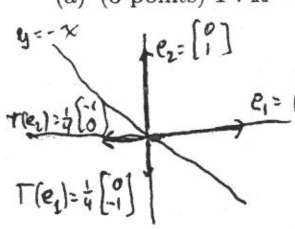
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 4 & -15 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_3-4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1-5R_3 \\ R_2+4R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 20 & -5 \\ 0 & 1 & 0 & 18 & -15 & 4 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right] \quad \therefore \quad M^{-1} = \begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$$

$$\text{Check } MM^{-1} = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

3. Find the matrix (with respect to the standard basis) for each one of the following linear transformations:

(a) (3 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection around the $y = -x$, followed by scaling by a factor of $1/4$.



$$[T] = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \left(= \frac{1}{4}\begin{bmatrix} \underset{\substack{\parallel \\ 0}}{\cos\frac{3\pi}{2}} & \underset{\substack{\parallel \\ -1}}{\sin\frac{3\pi}{2}} \\ \underset{\substack{\parallel \\ -1}}{\sin\frac{3\pi}{2}} & \underset{\substack{\parallel \\ 0}}{-\cos\frac{3\pi}{2}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right.$$

(b) (3 points) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection onto the xz-plane, followed by a scaling by a factor of 2.

$e_1 \mapsto e_1 \quad e_2 \mapsto 0 \quad e_3 \mapsto e_3$

$$[T] = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$.

You know that the matrix $B = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \; & r & \; & s & \; \end{array} \end{array}$ is the RREF of A.

(a) (3 points) Find a basis for Im A. Give reasons for your answer.

$$\boxed{\text{Basis for Im } A = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}}$$: columns of A with leading variables in B.

(b) (4 points) Find a basis for ker A.

Consider $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ s.t. $Av = \vec{0}$

Solving setting $x_2 = r$, $x_4 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r-5s \\ r \\ s \\ s \\ 0 \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Hence $$\boxed{\text{Basis for Ker } A = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

5. Consider the matrix $N = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} -5 & -6 \\ 1 & 0 \\ 1 & 6 \\ 3 & 0 \end{bmatrix} \end{array}$.

(a) (6 points) Use the Gram-Schmidt process to find a matrix Q with orthonormal columns such that $Im(Q) = Im(N)$.

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{(-5)^2+1^2+1^2+3^2}} v_1 = \frac{1}{6}\begin{bmatrix} -5 \\ 1 \\ 1 \\ 3 \end{bmatrix} \quad \Big\| \quad v_2 \cdot q_1 = \frac{1}{6}\begin{bmatrix} -6 \\ 0 \\ 6 \\ 0 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \\ 1 \\ 3 \end{bmatrix} = 6$$

$$v_2^{\perp} = v_2 - (v_2 \cdot q_1)q_1 = \begin{bmatrix} -6 \\ 0 \\ 6 \\ 0 \end{bmatrix} - 6\begin{bmatrix} -\frac{5}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 5 \\ -3 \end{bmatrix}$$

$$q_2 = \frac{v_2^{\perp}}{\|v_2^{\perp}\|} = \left(\frac{1}{\sqrt{(-1)^2+(-1)^2+5^2+(-3)^2}}\right) v_2^{\perp} = \frac{1}{6}\begin{bmatrix} -1 \\ -1 \\ 5 \\ -3 \end{bmatrix}$$

$$\text{So } Q = \begin{bmatrix} -\frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{5}{6} \\ \frac{3}{6} & -\frac{3}{6} \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{5}{6} \\ \frac{3}{6} & -\frac{3}{6} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -5 & -1 \\ 1 & -1 \\ 1 & 5 \\ 3 & -3 \end{bmatrix}$$

(b) (3 points) Using the matrix Q found in part (a), find a matrix R so that $N = QR$. Clearly label the matrix R.

$$R = \begin{bmatrix} v_1 \cdot q_1 & v_2 \cdot q_1 \\ v_1 \cdot q_2 & v_2 \cdot q_2 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 0 & 6 \end{bmatrix}$$

(c) (2 points) Find the matrix P of the orthogonal projection onto $Im(N)$.

$$P = QQ^T = \frac{1}{36}\begin{bmatrix} -5 & -1 \\ 1 & -1 \\ 1 & 5 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 1 & 3 \\ -1 & -1 & 5 & -3 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 26 & -4 & -10 & -12 \\ -4 & 2 & -4 & 6 \\ -10 & -4 & 26 & -12 \\ -12 & 6 & -12 & 18 \end{bmatrix}$$

(d) (1 point) Calculate the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ onto $Im(N)$.

$$Pv = \frac{1}{36}\begin{bmatrix} 26 & -4 & -10 & -12 \\ -4 & 2 & -4 & 6 \\ -10 & -4 & 26 & -12 \\ -12 & 6 & -12 & 18 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} -14 \\ -2 \\ 22 \\ -6 \end{bmatrix}$$

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6. (a) (8 points) Find the least–squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 6 \\ 0 \\ 6 \\ -4 \end{bmatrix},$$

$$A^T A \bar{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 6 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 36 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -36 \end{bmatrix} \Rightarrow \begin{array}{l} 4c_1 = 8 \\ c_1 = 2 \end{array} \quad \begin{array}{l} 36c_2 = -36 \\ c_2 = -1 \end{array}$$

$$\bar{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b) (2 points) Find the magnitude of the error of the least squares solution found in part (a).

$$E(\bar{x}) = \|b - A\bar{x}\| = \|b - Pb\| = \left\| \begin{bmatrix} 6 \\ 0 \\ 6 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 & -5 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 6 \\ 0 \\ 6 \\ -4 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ -1 \\ 5 \\ -3 \end{bmatrix} \right\| = \sqrt{1+1+5^2+3^2} = \boxed{6}$$

(c) (2 points) Using your work in part (a), find the equation of the line of the form $c_1 + c_2 x = y$ that best fits the points $(-5,\ 6)$, $(1,\ 0)$, $(1,\ 6)$ and $(3,\ -4)$ in the least squares sense.

Substituting the coordinates of the points into the equation of form $c_1 + c_2 x = y$ get:

$$\begin{array}{l} c_1 - 5c_2 = 6 \\ c_1 + c_2 = 0 \\ c_1 + c_2 = 6 \\ c_1 + 3c_2 = -4 \end{array} \Rightarrow \underbrace{\begin{bmatrix} 1 & -5 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ 6 \\ -4 \end{bmatrix}}_{b}$$

Least squares solution is $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Therefore $\boxed{2 - x = y}$ is line of best fit in least-squares sense.

SOLUTIONS

7. (5 points) Evaluate $det(L)$ if $L = \begin{bmatrix} 0 & 1 & 2 & c \\ 0 & 6 & 13 & c \\ 0 & 0 & 1 & b \\ a & 1 & -3 & c \end{bmatrix}$ in terms of a, b and c.

$$\det L = -a \det\begin{bmatrix} 1 & 2 & c \\ 6 & 13 & c \\ 0 & 1 & b \end{bmatrix} = -a\left(1 \det\begin{bmatrix} 13 & c \\ 1 & b \end{bmatrix} - 6\det\begin{bmatrix} 2 & c \\ 1 & b \end{bmatrix}\right)$$

$$= -a[(13b-c) - 6(2b-c)] = -a[b+5c]$$

Alt Sol

$$\det L = -\det\begin{bmatrix} a & 1 & -3 & c \\ 0 & 1 & 2 & c \\ 0 & 6 & 13 & c \\ 0 & 0 & 1 & b \end{bmatrix} = -\det\begin{bmatrix} a & 1 & -3 & c \\ 0 & 1 & 2 & c \\ 0 & 0 & 1 & -5c \\ 0 & 0 & 1 & b \end{bmatrix} R_3 - 6R_2$$

$$= -\det\begin{bmatrix} a & 1 & -3 & c \\ 0 & 1 & 2 & c \\ 0 & 0 & 1 & -5c \\ 0 & 0 & 0 & (b+5c) \end{bmatrix} R_4 - R_3 = -(a)(1)(1)(b+5c)$$

$$= \boxed{-a(b+5c)}$$

8. (5 points) Suppose that A, B and C are 4×4 matrices such that $det(A) = x \neq 0$, $det(B) = y \neq 0$ and $det(C) = z \neq 0$. Compute the value of the following determinant in terms of x, y and z.

$$det[3(A^4B^4)^{-1}(BC)^2(AB)^T]$$

$$= 3^4 \det(A^4B^4)^{-1} \det(BC)^2 \det(AB)^T$$

$$= 3^4 \frac{1}{(\det A)^4(\det B)^4} (\det B)^2(\det C)^2(\det A)(\det B)$$

$$= \frac{3^4}{x^4y^4} y^2z^2xy$$

$$= \boxed{\frac{3^4 z^2}{x^3 y}}$$

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9. Let $A = \begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

(a) (3 points) Find the characteristic polynomial of A.

$$\det(A-\lambda I) = \det\begin{bmatrix} (3-\lambda) & 5 & -2 \\ 0 & (2-\lambda) & 0 \\ 0 & 2 & (1-\lambda) \end{bmatrix} = (3-\lambda)\det\begin{bmatrix} (2-\lambda) & 0 \\ 2 & (1-\lambda) \end{bmatrix}$$

$$= \boxed{(3-\lambda)(2-\lambda)(1-\lambda)}$$

(b) (3 points) Find the eigenvalues of A.

set $\det(A-\lambda I)=0 \Rightarrow \boxed{\lambda = 3, 2, 1}$ are the eigenvalues

(c) (6 points) Find a basis for each of the eigenspaces of A

$\underline{\lambda=3}$ $\quad (A-3I)v = \begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a=1,\ b=0,\ c=0,\quad E_3 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

basis

$\underline{\lambda=2}$ $\quad (A-2I)v = \begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a=-1,\ b=1,\ c=2 \quad E_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right\}$

$\underline{\lambda=1}$ $\quad (A-I)v = \begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a=1,\ b=0,\ c=1 \quad E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

basis

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11. Consider $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Let $M = A^T A$.

(a) (2 points) Find M.

$$A^T A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M$$

(b) (3 points) Show that the following vectors are eigenvectors for M and determine the corresponding eigenvalues.

$$\vec{b_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \vec{b_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \vec{b_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$M b_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \lambda = 3 \text{ and } E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$M b_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \lambda = 1 \text{ and } E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$M b_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \lambda = 0 \text{ and } E_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) (3 points) Find an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ for $\mathbb{R}^3$ consisting of eigenvectors for M.

$$\text{orthonormal basis } \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$$

(d) (1 point) Find the singular values of A.

$$\sigma = \sqrt{3}, 1, 0$$

SOLUTIONS

(Continuation of problem 9 from previous page)

(d) (2 points) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$. **YOU NEED NOT COMPUTE** S^{-1}.

$$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 6 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Then } A = SDS^{-1}$$

10. Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + 2x_1x_3 + 2x_2^2 + x_3^2$$

(a) (3 points) Find the matrix A of q.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) (5 points) Determine the definiteness of q (equivalently, the definiteness of A). Explain your reasoning.

Find eigenvalues of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (1-\lambda) & 0 & 1 \\ 0 & (2-\lambda) & 0 \\ 1 & 0 & (1-\lambda) \end{bmatrix}$$

$$= (2-\lambda)\det\begin{bmatrix} (1-\lambda) & 1 \\ 1 & (1-\lambda) \end{bmatrix} = (2-\lambda)\left[(1-\lambda)(1-\lambda) - 1\right]$$

$$= (2-\lambda)\left[\lambda^2 - 2\lambda + 1 - 1\right] = (2-\lambda)(\lambda)(\lambda-2) = 0$$

Eigenvalues are $\lambda = 2, 2, 0$

A is symmetric with orthonormal eigenbasis v_1, v_2, v_3. When $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is written in this basis as $x = c_1 v_1 + c_2 v_2 + c_3 v_3$ so that $x_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ the form is diagonalized as $q(x) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$.

As λ_1 and $\lambda_2 > 0$ and $\lambda_3 = 0$, the values of $q(x) \geq 0$ $\forall x$.

Hence q is $\boxed{\text{positive semidefinite}}$

SOLUTIONS

(Continuation of problem 11 from previous page)

(e) (2 points) Determine the vectors $A\vec{v_1}, A\vec{v_2}, A\vec{v_3}$.

$$Av_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{6} \\ 3/\sqrt{6} \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$Av_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(f) (3 points) Using your work in (a)-(e), determine the matrices V, Σ, U of the singular value decomposition of A (you do not need to multiply U, Σ and V^T). Label your answers clearly.

$$A = U \Sigma V^T$$

where $U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$ and $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 3/\sqrt{6} \\ 3/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{18} \\ 3/\sqrt{18} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Hence $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$