**NORTHEASTERN UNIVERSITY**
**Department of Mathematics**

**MATH 2331 (Linear Algebra)** **Final Exam — Fall 2016**

Do not write in these boxes:

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Score | | | | | | | | | | | | |
| Max Pts | 10 | 7 | 6 | 7 | 12 | 12 | 5 | 5 | 14 | 8 | 14 | 100 |

**Name:** ______________________ **Instructor:** ______________________

**Instructions:**

- Write your name and your instructor's name in the blanks above.
- **SHOW YOUR WORK.** If there is not enough room to show your work, use the back of the preceding page.
- **Sufficient work must be shown to justify answers. No calculator or any other reference is permitted.**

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1. (10 points) Consider the system:

$$\begin{array}{rcrcrcrcrcrcr} 2x_1 & + & 4x_2 & + & x_3 & + & x_4 & - & x_5 & + & x_6 & = & 5 \\ x_1 & + & 2x_2 & + & x_3 & + & x_4 & - & x_5 & + & x_6 & = & -3 \\ & & & & & & x_4 & - & x_5 & & & = & 2 \\ & & & & & - & x_4 & + & x_5 & + & x_6 & = & 1 \end{array}$$

Find all solutions of the system using Gauss-Jordan elimination on the augmented matrix of the system. Be sure to indicate which variables, if any, are free. Show all steps of your computations.

2. (7 points) Given the matrix $M = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$. Use Gauss-Jordan elimination to compute the inverse of $M$.

3. Find the matrix (with respect to the standard basis) for each one of the following linear transformations:

   (a) (3 points) $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection around the $y = -x$, followed by scaling by a factor of $1/4$.

   (b) (3 points) $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection onto the $xz$-plane, followed by a scaling by a factor of 2.

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$.

   You know that the matrix $B = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the RREF of $A$.

   (a) (3 points) Find a basis for $\operatorname{Im} A$. Give reasons for your answer.

   (b) (4 points) Find a basis for $\ker A$.

5. Consider the matrix $N = \begin{bmatrix} -5 & -6 \\ 1 & 0 \\ 1 & 6 \\ 3 & 0 \end{bmatrix}$.

   (a) (6 points) Use the Gram-Schmidt process to find a matrix $Q$ with orthonormal columns such that $Im(Q) = Im(N)$.

   (b) (3 points) Using the matrix $Q$ found in part (a), find a matrix $R$ so that $N = QR$. Clearly label the matrix $R$.

   (c) (2 points) Find the matrix $P$ of the orthogonal projection onto $Im(N)$.

   (d) (1 point) Calculate the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ onto $Im(N)$.

6. (a) (8 points) Find the least–squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 6 \\ 0 \\ 6 \\ -4 \end{bmatrix},$$

(b) (2 points) Find the magnitude of the error of the least squares solution found in part (a).

(c) (2 points) Using your work in part (a), find the equation of the line of the form $c_1 + c_2x = y$ that best fits the points $(-5,\ 6)$, $(1,\ 0)$, $(1,\ 6)$ and $(3,\ -4)$ in the least squares sense.

7. (5 points) Evaluate $det(L)$ if $L = \begin{bmatrix} 0 & 1 & 2 & c \\ 0 & 6 & 13 & c \\ 0 & 0 & 1 & b \\ a & 1 & -3 & c \end{bmatrix}$ in terms of $a$, $b$ and $c$.

8. (5 points) Suppose that $A$, $B$ and $C$ are $4 \times 4$ matrices such that $det(A) = x \neq 0$, $det(B) = y \neq 0$ and $det(C) = z \neq 0$. Compute the value of the following determinant in terms of $x$, $y$ and $z$.

$$det[3(A^4B^4)^{-1}(BC)^2(AB)^T]$$

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9. Let $A = \begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

(a) (3 points) Find the characteristic polynomial of $A$.

(b) (3 points) Find the eigenvalues of $A$.

(c) (6 points) Find a basis for each of the eigenspaces of $A$.

(Continuation of problem 9 from previous page)

(d) (2 points) Find an invertible matrix $S$ and a diagonal matrix $D$ such that $A = SDS^{-1}$. **YOU NEED NOT COMPUTE $S^{-1}$.**

10. Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + 2x_1x_3 + 2x_2^2 + x_3^2$$

(a) (3 points) Find the matrix $A$ of $q$.

(b) (5 points) Determine the definiteness of $q$ (equivalently, the definiteness of $A$). Explain your reasoning.

11. Consider $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Let $M = A^T A$.

(a) (2 points) Find $M$.

(b) (3 points) Show that the following vectors are eigenvectors for $M$ and determine the corresponding eigenvalues.

$$\vec{b_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ \vec{b_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ \vec{b_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(c) (3 points) Find an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ for $\mathbb{R}^3$ consisting of eigenvectors for $M$.

(d) (1 point) Find the singular values of $A$.

(Continuation of problem 11 from previous page)

(e) (2 points) Determine the vectors $A\vec{v_1}, A\vec{v_2}, A\vec{v_3}$.

(f) (3 points) Using your work in (a)-(e), determine the matrices $V, \Sigma, U$ of the singular value decomposition of $A$ (you do not need to multiply $U$, $\Sigma$ and $V^T$). Label your answers clearly.