Math 2331 (Linear Algebra)

Final Exam Review $\sim$ December 10th, 2021

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Final Exam Topics

The topics for the final exam are as follows:

- Systems of linear equations, row-reduction, echelon forms
- Matrix algebra, inverse matrices, determinants, properties of determinants
- Subspaces of $\mathbb{R}^n$, linear independence, span, bases of subspaces, row space/column space/nullspace
- Linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$, kernel and image, associated matrices, coordinate vectors, change of basis
- The dot product in $\mathbb{R}^n$, lengths and angles, orthogonal and orthonormal sets and bases, Gram-Schmidt, QR factorization, orthogonal complements and orthogonal projections, least squares
- Eigenvalues and eigenvectors, characteristic polynomials, properties of eigenvalues, diagonalization, orthogonal matrices and the real spectral theorem, computing matrix powers
- Quadratic forms on $\mathbb{R}^n$, associated matrices, definiteness
- Singular values and singular value decompositions of matrices

Final Exam Information

Other brief pieces of information about the final exam:

- The final exam is all free response (no true/false or multiple choice) and is approximately 10 pages in length.
- The exam format is similar to the old final exams.
- Calculators are permitted, but all relevant work including appropriate intermediate steps must be shown. Points may be deducted for calculator usage that trivializes a problem when no other work is shown (for example, using a calculator to compute a determinant when the computation of the determinant is the entire problem, or using it to solve a system of equations when finding the solution is the entire problem).
- You may bring a 1-page, 8.5in-by-11in note sheet to the exam. You may write or type anything you want on both sides of the note sheet.
- The official exam time limit is 2 hours (120 minutes).

Review Problems, I

(Fa20-#1a) Solve the linear system $\begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 = 3 \\ 2x_3 - 8x_4 = 2 \\ 2x_1 + 12x_2 + 2x_3 - 2x_4 = 4 \end{cases}$ by elementary row operations and write the parametric vector form for all solutions.

Review Problems, I

(Fa20-#1a) Solve the linear system $\begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 = 3 \\ 2x_3 - 8x_4 = 2 \\ 2x_1 + 12x_2 + 2x_3 - 2x_4 = 4 \end{cases}$ by elementary row operations and write the parametric vector form for all solutions.

- We row-reduce the coefficient matrix:
$$\left[\begin{array}{cccc|c} 1 & 6 & 2 & -5 & 3 \\ 0 & 0 & 2 & -8 & 2 \\ 2 & 12 & 2 & -2 & 4 \end{array}\right] \xrightarrow[R_3/2]{R_2/2} \left[\begin{array}{cccc|c} 1 & 6 & 2 & -5 & 3 \\ 0 & 0 & 1 & -4 & 1 \\ 1 & 6 & 1 & -1 & 2 \end{array}\right]$$
$$\xrightarrow[R_3+R_2]{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 6 & 2 & -5 & 3 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cccc|c} 1 & 6 & 0 & 3 & 1 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
- From the RREF we get free variables x_2, x_4 and so taking $x_2 = a$, $x_4 = b$ we get the solution $(x_1, x_2, x_3, x_4) = (-6a - 3b + 1, a, 4b + 1, b) =$ $\boxed{(1, 0, 1, 0) + a(-6, 1, 0, 0) + b(-3, 0, 4, 1)}$.

Review Problems, II

(Fa20-#1b) Let $A = \begin{bmatrix} 1 & 6 & 2 & -5 \\ 0 & 0 & 2 & -8 \\ 2 & 12 & 2 & -2 \end{bmatrix}$; note $\begin{bmatrix} 1 & 6 & 0 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the RREF of A. Find bases for $\ker(A)$, $\mathrm{im}(A)$, and $[\mathrm{im}(A^T)]^\perp$.

Review Problems, II

(Fa20-#1b) Let $A = \begin{bmatrix} 1 & 6 & 2 & -5 \\ 0 & 0 & 2 & -8 \\ 2 & 12 & 2 & -2 \end{bmatrix}$; note $\begin{bmatrix} 1 & 6 & 0 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the RREF of A. Find bases for $\ker(A)$, $\mathrm{im}(A)$, and $[\mathrm{im}(A^T)]^\perp$.

- For $\ker(A)$, aka the nullspace, is the solution set to the homogeneous system $x_1 + 6x_2 + 3x_4 = 0$, $x_3 - 4x_4 = 0$. The solutions are $(x_1, x_2, x_3, x_4) = (-6a - 3b, a, 4b, b) = a(-6, 1, 0, 0) + b(-3, 0, 4, 1)$, so we get a basis $\boxed{(-6, 1, 0, 0), (-3, 0, 4, 1)}$.
- For $\mathrm{im}(A)$, aka the column space, we take the columns of A that have pivots in the RREF. These are the first and third columns, yielding a basis $\boxed{(1, 0, 2), (2, 2, 2)}$.
- For $[\mathrm{im}(A^T)]^\perp$, note that $\mathrm{im}(A^T)$ is the row space of A, and so its orthogonal complement is the nullspace of A, whose basis we found above is $\boxed{(-6, 1, 0, 0), (-3, 0, 4, 1)}$.

Review Problems, III

(Fa20-#2) Use row operations to find the inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Review Problems, III

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- We set up the double matrix and then row-reduce the left:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -4 & -3 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -4 & -3 & -2 & 1 & 0 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right] \xrightarrow[R_1-2R_3]{R_2-R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 5 & -2 & -8 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right]$$

- So the inverse is $\boxed{\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & -3 \\ -2 & 1 & 4 \end{bmatrix}}$.

Review Problems, IV

(Fa20-#3) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that begins with reflection about the y-axis, followed by orthogonal projection onto the line $y = 3x$. Find the matrix of the transformation T.

Review Problems, IV

(Fa20-#3) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that begins with reflection about the y-axis, followed by orthogonal projection onto the line $y = 3x$. Find the matrix of the transformation T.

- The matrix for reflection about the y-axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ since it negates x-coordinates and preserves y-coordinates. (Or we could use the formula for reflection across an arbitrary line.)
- The matrix for projection onto $y = 3x$ is $\frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$.
- The matrix for the desired composition is then the product $\frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1/10 & 3/10 \\ -3/10 & 9/10 \end{bmatrix}}$.

Review Problems, V

(Fa20-#4) Let $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ be a subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

1. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal basis for V?
2. Find the decomposition of $\mathbf{y} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ as $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ where $\mathbf{y}_1 \in V$ and $\mathbf{y}_2 \in V^{\perp}$.
3. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by orthogonal projection onto V. Find the matrix of the linear transformation T.

[On the next slide I switch to row vectors for space reasons!]

Review Problems, VI

(Fa20-#4) Let $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1/\sqrt{2}, 0, -1/\sqrt{2}, 0)$, $\mathbf{v}_2 = (1/2, 1/2, 1/2, 1/2)$.

1. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal basis for V?

Review Problems, VI

(Fa20-#4) Let $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1/\sqrt{2}, 0, -1/\sqrt{2}, 0)$, $\mathbf{v}_2 = (1/2, 1/2, 1/2, 1/2)$.

1. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal basis for V?

- $\boxed{\text{Yes}}$: we have $||\mathbf{v}_1|| = 1$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, and $||\mathbf{v}_1|| = 1$.

2. Find the decomposition of $\mathbf{y} = (4, -1, 0, 1)$ as $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ where $\mathbf{y}_1 \in V$ and $\mathbf{y}_2 \in V^{\perp}$.

Review Problems, VI

(Fa20-#4) Let $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1/\sqrt{2}, 0, -1/\sqrt{2}, 0)$, $\mathbf{v}_2 = (1/2, 1/2, 1/2, 1/2)$.

1. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal basis for V?

- $\boxed{\text{Yes}}$: we have $||\mathbf{v}_1|| = 1$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, and $||\mathbf{v}_1|| = 1$.

2. Find the decomposition of $\mathbf{y} = (4, -1, 0, 1)$ as $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ where $\mathbf{y}_1 \in V$ and $\mathbf{y}_2 \in V^\perp$.

- The vector $\mathbf{y}_1$ is the orthogonal projection of $\mathbf{y}$ into V, which since $\mathbf{v}_1, \mathbf{v}_2$ are an orthonormal basis of V, is $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ where $a_1 = \mathbf{y} \cdot \mathbf{v}_1 = 2\sqrt{2}$ and $a_2 = \mathbf{y} \cdot \mathbf{v}_2 = 2$.
- So $\mathbf{y}_1 = 2\sqrt{2}\mathbf{v}_1 + 2\mathbf{v}_2 = \boxed{(3, 1, -1, 1)}$.
- Then $\mathbf{y}_2 = \mathbf{y} - \mathbf{y}_1 = \boxed{(1, -2, 1, 0)}$. As a check, $\mathbf{y}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so it is in $V^\perp$.

Review Problems, VII

(Fa20-#4) Let $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1/\sqrt{2}, 0, -1/\sqrt{2}, 0)$, $\mathbf{v}_2 = (1/2, 1/2, 1/2, 1/2)$.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by orthogonal projection onto V. Find the matrix of the linear transformation T.

Review Problems, VII

(Fa20-#4) Let $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1/\sqrt{2}, 0, -1/\sqrt{2}, 0)$, $\mathbf{v}_2 = (1/2, 1/2, 1/2, 1/2)$.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by orthogonal projection onto V. Find the matrix of the linear transformation T.

- If A has columns $\mathbf{v}_1$, $\mathbf{v}_2$, the projection onto the image (column space) of A is $P = A(A^T A)^{-1} A^T$.
- With $A = \begin{bmatrix} 1/\sqrt{2} & 1/2 \\ 0 & 1/2 \\ -1/\sqrt{2} & 1/2 \\ 0 & 1/2 \end{bmatrix}$ we have $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

so $P = AA^T = \boxed{\dfrac{1}{4}\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$.

Review Problems, VIII

(Fa20-#5) Suppose $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ form a basis for the vector space W.

1. Use Gram-Schmidt to find an orthogonal basis for W.
2. Normalize the result above to give an orthonormal basis for W.
3. Find the QR factorization of $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

[Again, I'll use row vectors on the next slides.]

Review Problems, IX

(Fa20-#5) Suppose $\mathbf{x}_1 = (1, 0, 1)$,$\mathbf{x}_2 = (3, 1, 1)$, $\mathbf{x}_3 = (3, 4, 1)$ form a basis for the vector space W.

1. Use Gram-Schmidt to find an orthogonal basis for W.

Review Problems, IX

(Fa20-#5) Suppose $\mathbf{x}_1 = (1, 0, 1)$,$\mathbf{x}_2 = (3, 1, 1)$, $\mathbf{x}_3 = (3, 4, 1)$ form a basis for the vector space W.

1. Use Gram-Schmidt to find an orthogonal basis for W.

- First we take $\mathbf{w}_1 = \mathbf{x}_1 = \boxed{(1, 0, 1)}$.
- Next, $\mathbf{w}_2 = \mathbf{x_2} - a_1\mathbf{w}_1$ where $a_1 = \dfrac{\mathbf{x}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{4}{2} = 2$, so
  $\mathbf{w}_2 = (3, 1, 1) - 2(1, 0, 1) = \boxed{(1, 1, -1)}$.
- Finally, $\mathbf{w}_3 = \mathbf{x_3} - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\mathbf{x}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{4}{2} = 2$
  and $b_2 = \dfrac{\mathbf{x}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} = \dfrac{6}{3} = 2$ so
  $\mathbf{w}_2 = (3, 4, 1) - 2(1, 0, 1) - 2(1, 1, -1) = \boxed{(-1, 2, 1)}$.

2. Normalize the result to give an orthonormal basis for W.

Review Problems, IX

(Fa20-#5) Suppose $\mathbf{x}_1 = (1,0,1)$,$\mathbf{x}_2 = (3,1,1)$, $\mathbf{x}_3 = (3,4,1)$ form a basis for the vector space W.

1. Use Gram-Schmidt to find an orthogonal basis for W.

- First we take $\mathbf{w}_1 = \mathbf{x}_1 = \boxed{(1,0,1)}$.
- Next, $\mathbf{w}_2 = \mathbf{x_2} - a_1\mathbf{w}_1$ where $a_1 = \dfrac{\mathbf{x}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{4}{2} = 2$, so $\mathbf{w}_2 = (3,1,1) - 2(1,0,1) = \boxed{(1,1,-1)}$.
- Finally, $\mathbf{w}_3 = \mathbf{x_3} - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\mathbf{x}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{4}{2} = 2$ and $b_2 = \dfrac{\mathbf{x}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} = \dfrac{6}{3} = 2$ so $\mathbf{w}_2 = (3,4,1) - 2(1,0,1) - 2(1,1,-1) = \boxed{(-1,2,1)}$.

2. Normalize the result to give an orthonormal basis for W.

- This gives $\boxed{(1,0,1)/\sqrt{2},\ (1,1,-1)/\sqrt{3},\ (-1,2,1)/\sqrt{6}}$.

Review Problems, X

(Fa20-#5) Suppose $\mathbf{x}_1 = (1, 0, 1)$,$\mathbf{x}_2 = (3, 1, 1)$, $\mathbf{x}_3 = (3, 4, 1)$ form a basis for the vector space W.

3. Find the QR factorization of $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

Review Problems, X

(Fa20-#5) Suppose $\mathbf{x}_1 = (1, 0, 1)$,$\mathbf{x}_2 = (3, 1, 1)$, $\mathbf{x}_3 = (3, 4, 1)$ form a basis for the vector space W.

3. Find the QR factorization of $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

- The columns of Q (orthogonal) are given by the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we calculated, while entries of R (upper-triangular) are given by the inner products $\mathbf{e}_i \cdot \mathbf{x}_j$.
- Explicitly, $r_{1,1} = ||\mathbf{w}_1|| = \sqrt{2}$, $r_{1,2} = a_1||\mathbf{w}_1|| = 2\sqrt{2}$, $r_{1,3} = b_1||\mathbf{w}_1|| = 2\sqrt{2}$, $r_{2,2} = ||\mathbf{w}_2|| = \sqrt{3}$, $r_{2,3} = b_2||\mathbf{w}_2|| = 2\sqrt{3}$, $r_{3,3} = ||\mathbf{w}_3|| = \sqrt{6}$. Thus,

$$Q = \boxed{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}}, \quad R = \boxed{\begin{bmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{3} & 2\sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}}.$$

Review Problems, XI

(Fa20-#6) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

1. Find the least-squares solution $\hat{\mathbf{x}}$ of the system $A\mathbf{x} = \mathbf{b}$.

Review Problems, XI

(Fa20-#6) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

1. Find the least-squares solution $\hat{\mathbf{x}}$ of the system $A\mathbf{x} = \mathbf{b}$.

- Since the columns of A are linearly independent, the least-squares solution is given by $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$.
- Here $A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$ so

$$\hat{\mathbf{x}} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}.$$

Review Problems, XII

(Fa20-#6) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

2. The image $\mathrm{im}(A)$ is a plane in $\mathbb{R}^3$ passing through the origin. Find the distance from $\mathbf{b} = (1, 2, 3)$ to the plane $\mathrm{im}(A)$.

Review Problems, XII

(Fa20-#6) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

2. The image $\mathrm{im}(A)$ is a plane in $\mathbb{R}^3$ passing through the origin. Find the distance from $\mathbf{b} = (1, 2, 3)$ to the plane $\mathrm{im}(A)$.

- The projection of $\mathbf{b}$ into the column space of A is the vector
$$A\hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$
- Therefore, the desired distance is the distance between $\mathbf{b}$ and the projection, which is the length of the vector $(1, 2, 3) - (2, 2, 2) = (-1, 0, 1)$, so the distance is $\boxed{\sqrt{2}}$.

Review Problems, XIII

(Fa20-#7) Consider the transition matrix $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$.

1. Calculate an eigenvector of A of eigenvalue 1.

Review Problems, XIII

(Fa20-#7) Consider the transition matrix $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$.

1. Calculate an eigenvector of A of eigenvalue 1.

- This is a vector in the kernel of $I_2 - A$

$$= \begin{bmatrix} 0.5 & -0.2 \\ -0.5 & 0.2 \end{bmatrix} \xrightarrow[2 \cdot R_1]{R_2 + R_1} \begin{bmatrix} 1 & -0.4 \\ 0 & 0 \end{bmatrix}, \text{ giving } \mathbf{v} = \boxed{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}.$$

2. Find $\lim_{n \to \infty} A^n$.

Review Problems, XIII

(Fa20-#7) Consider the transition matrix $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$.

1. Calculate an eigenvector of A of eigenvalue 1.

- This is a vector in the kernel of $I_2 - A$ $= \begin{bmatrix} 0.5 & -0.2 \\ -0.5 & 0.2 \end{bmatrix} \xrightarrow[2 \cdot R_1]{R_2 + R_1} \begin{bmatrix} 1 & -0.4 \\ 0 & 0 \end{bmatrix}$, giving $\mathbf{v} = \boxed{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}$.

2. Find $\lim_{n \to \infty} A^n$.

- From the general theory of Markov chains, since this is a stochastic matrix (entries nonnegative, columns sum to 1), all columns of the limit will be the 1-eigenvector whose column sum is 1. This matrix is $\boxed{\begin{bmatrix} 2/7 & 2/7 \\ 5/7 & 5/7 \end{bmatrix}}$. (One can also diagonalize A and then take the limit to find this.)

Review Problems, XIV

(Fa20-#8) Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 8 \\ 1 & 5 & 7 & 9 \\ 0 & 0 & 0 & 2 \\ 2 & 5 & 8 & 10 \end{bmatrix}$.

1. Calculate the determinant of A.

Review Problems, XIV

(Fa20-#8) Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 8 \\ 1 & 5 & 7 & 9 \\ 0 & 0 & 0 & 2 \\ 2 & 5 & 8 & 10 \end{bmatrix}$.

1. Calculate the determinant of A.

- Use row operations + expanding along empty rows/columns:

$$\begin{vmatrix} 1 & 2 & 3 & 8 \\ 1 & 5 & 7 & 9 \\ 0 & 0 & 0 & 2 \\ 2 & 5 & 8 & 10 \end{vmatrix} \underset{R_4-2R_1}{\overset{R_2-R_1}{=}} \begin{vmatrix} 1 & 2 & 3 & 8 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & -6 \end{vmatrix} \overset{C_1}{=} 1 \cdot \begin{vmatrix} 3 & 4 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & -6 \end{vmatrix}$$

$$\overset{R_2}{=} 1 \cdot (-2) \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 1 \cdot (-2) \cdot 2 = \boxed{-4}.$$

Review Problems, XV

(Fa20-#8) Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 8 \\ 1 & 5 & 7 & 9 \\ 0 & 0 & 0 & 2 \\ 2 & 5 & 8 & 10 \end{bmatrix}$.

2. Find $\det[(4A)^{-1}(2A^T)^2]$.

Review Problems, XV

(Fa20-#8) Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 8 \\ 1 & 5 & 7 & 9 \\ 0 & 0 & 0 & 2 \\ 2 & 5 & 8 & 10 \end{bmatrix}$.

2. Find $\det[(4A)^{-1}(2A^T)^2]$.

- Since A is 4×4, we have $\det(4A) = 4^4 \det(A) = 4^4(-4) = -1024$ so $\det(4A)^{-1} = \dfrac{1}{\det(A)} = -\dfrac{1}{1024}$.
- Also, $\det(2A^T) = 2^4 \det(A^T) = 2^4 \det(A) = 2^4(-4) = -64$, so $\det[(2A^T)^2] = (-64)^2 = 4096$.
- Finally, $\det[(4A)^{-1}(2A^T)^2] = \det(4A)^{-1} \det[(2A^T)^2] = \boxed{-4}$.

Review Problems, XVII

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

1. Find a basis for the eigenspace E_λ with eigenvalue $\lambda_1 = 3$.

Review Problems, XVII

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

1. Find a basis for the eigenspace E_λ with eigenvalue $\lambda_1 = 3$.

- The 3-eigenspace is the nullspace (kernel) of
$$3I_3 - A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -1 & -2 \\ 1 & -1 & -2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
- So the 3-eigenspace is 2-dimensional.
- Solving the system $a - b - 2c = 0$ yields $(a, b, c) = (b + 2c, b, c) = b(1, 1, 0) + c(2, 0, 1)$ so we get the basis $\boxed{\{(1, 1, 0),\ (2, 0, 1)\}}$.

Review Problems, XVIII

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

2. What are the geometric multiplicities of $\lambda_1 = 3$ and $\lambda_2 = 5$?

Review Problems, XVIII

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

2. What are the geometric multiplicities of $\lambda_1 = 3$ and $\lambda_2 = 5$?

- The geometric multiplicity is simply the dimension of the corresponding eigenspace. The three eigenvalues of A are $\lambda = 3, 3, 5$ [note that the trace of A is 11, so the other eigenvalue is also 3].
- So the geometric multiplicity of $\lambda = 3$ is $\boxed{2}$ by the calculation just made, and the geometric multiplicity of $\lambda = 5$ can only be $\boxed{1}$ since its algebraic multiplicity is 1 (i.e., it is only a single root of the characteristic polynomial).

Review Problems, XIX

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

3. Is A diagonalizable? Explain.

Review Problems, XIX

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

3. Is A diagonalizable? Explain.

- Since all of the eigenspaces have full dimension, $\boxed{A \text{ is diagonalizable}}$. (To find it, we would also need to find a 5-eigenvector, which one may check is $(1, 1, 1)$ so $A = QDQ^{-1}$ with $Q = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.)

4. Is A orthogonally diagonalizable? Explain.

Review Problems, XIX

(Fa20-#9) The matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ has distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$.

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- Since all of the eigenspaces have full dimension, $\boxed{A \text{ is diagonalizable}}$. (To find it, we would also need to find a 5-eigenvector, which one may check is $(1, 1, 1)$ so $A = QDQ^{-1}$ with $Q = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.)

4. Is A orthogonally diagonalizable? Explain.

- $\boxed{\text{No}}$: if $A = QDQ^T$ where $Q^{-1} = Q^T$ is orthogonal, then $A^T = (QDQ^T)^T = QD^TQ^T = QDQ^T = A$ so A would be symmetric (which it isn't).

Review Problems, XX

(Fa20-#10) Consider $A = \begin{bmatrix} -1 & 5 & 7 \\ 3 & -3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$.

1. Find all eigenvalues of A.

Review Problems, XX

(Fa20-#10) Consider $A = \begin{bmatrix} -1 & 5 & 7 \\ 3 & -3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$.

1. Find all eigenvalues of A.

- The characteristic polynomial is $\det(tI_3 - A) =$
$\begin{vmatrix} t+1 & -5 & -7 \\ -3 & t+3 & -8 \\ 0 & 0 & t-2 \end{vmatrix} \overset{R_3}{=} (t-2) \begin{vmatrix} t+1 & -5 \\ -3 & t+3 \end{vmatrix} =$
$(t-2)(t^2+4t-12) = (t-2)^2(t+6)$.
- So the eigenvalues are $\lambda = \boxed{2, 2, -6}$.

2. What are the eigenvalues of $A^2 - 3A$?

Review Problems, XX

(Fa20-#10) Consider $A = \begin{bmatrix} -1 & 5 & 7 \\ 3 & -3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$.

1. Find all eigenvalues of A.

- The characteristic polynomial is $\det(tI_3 - A) =$
$\begin{vmatrix} t+1 & -5 & -7 \\ -3 & t+3 & -8 \\ 0 & 0 & t-2 \end{vmatrix} \overset{R_3}{=} (t-2)\begin{vmatrix} t+1 & -5 \\ -3 & t+3 \end{vmatrix} =$
$(t-2)(t^2+4t-12) = (t-2)^2(t+6)$.
- So the eigenvalues are $\lambda = \boxed{2, 2, -6}$.

2. What are the eigenvalues of $A^2 - 3A$?

- For any polynomial q, the eigenvalues of $q(A)$ are the values $q(\lambda)$ for the eigenvalues λ. Plugging in gives $\boxed{-2, -2, 54}$.

Review Problems, XXI

(Fa20-#11) The symmetric matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ has eigenvectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ with corresponding eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$, $\lambda_3 = 0$.

1. Orthogonally diagonalize the matrix A.

Review Problems, XXI

(Fa20-#11) The symmetric matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ has eigenvectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ with corresponding eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$, $\lambda_3 = 0$.

1. Orthogonally diagonalize the matrix A.

- We just normalize the eigenvectors to get columns of U and give the corresponding eigenvalues for D:

$$U = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}, \; D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Review Problems, XXII

(Fa20-#11) The symmetric matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ has an orthogonal matrix $U = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$.

2. Let $\mathbf{a} = \begin{bmatrix} \sqrt{5} \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \sqrt{5} \\ \sqrt{3} \\ 0 \end{bmatrix}$. Find $||U\mathbf{a}||$ and $U\mathbf{a} \cdot U\mathbf{b}$.

Review Problems, XXII

(Fa20-#11) The symmetric matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ has an orthogonal matrix $U = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$.

2. Let $\mathbf{a} = \begin{bmatrix} \sqrt{5} \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \sqrt{5} \\ \sqrt{3} \\ 0 \end{bmatrix}$. Find $||U\mathbf{a}||$ and $U\mathbf{a} \cdot U\mathbf{b}$.

- U is orthogonal so it preserves lengths and dot products. So $||U\mathbf{a}|| = ||\mathbf{a}|| = \boxed{\sqrt{11}}$ and $U\mathbf{a} \cdot U\mathbf{b} = \mathbf{a} \cdot \mathbf{b} = 5 + 3 + 0 = \boxed{8}$.

Review Problems, XXIII

(Fa20-#12) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

1. Compute $B = A^T A$.

Review Problems, XXIII

(Fa20-#12) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

1. Compute $B = A^T A$.

- $B = \begin{bmatrix} 0 & 3 \\ 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 0 & 12 \\ 0 & 16 & 0 \\ 12 & 0 & 16 \end{bmatrix}}$.

2. Verify that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ are eigenvectors of B and find their eigenvalues.

Review Problems, XXIII

(Fa20-#12) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

1. Compute $B = A^T A$.

- $B = \begin{bmatrix} 0 & 3 \\ 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 0 & 12 \\ 0 & 16 & 0 \\ 12 & 0 & 16 \end{bmatrix}}$.

2. Verify that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ are eigenvectors of B and find their eigenvalues.

- We have $B\mathbf{v}_1 = 25\mathbf{v}_1$ (so $\lambda = \boxed{25}$), $B\mathbf{v}_2 = 16\mathbf{v}_2$ (so $\lambda = \boxed{16}$), $B\mathbf{v}_3 = 0\mathbf{v}_3$ (so $\lambda = \boxed{0}$).

3. What are the singular values of A?

Review Problems, XXIII

(Fa20-#12) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

1. Compute $B = A^T A$.
   - $B = \begin{bmatrix} 0 & 3 \\ 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 0 & 12 \\ 0 & 16 & 0 \\ 12 & 0 & 16 \end{bmatrix}}$.
2. Verify that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ are eigenvectors of B and find their eigenvalues.
   - We have $B\mathbf{v}_1 = 25\mathbf{v}_1$ (so $\lambda = \boxed{25}$), $B\mathbf{v}_2 = 16\mathbf{v}_2$ (so $\lambda = \boxed{16}$), $B\mathbf{v}_3 = 0\mathbf{v}_3$ (so $\lambda = \boxed{0}$).
3. What are the singular values of A?
   - These are the square roots of the positive eigenvalues of $A^T A$ so $\sigma_1 = \boxed{5}$, $\sigma_2 = \boxed{4}$.

Review Problems, XXIV

(Fa20-#12) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

4. Find the singular value decomposition $A = U\Sigma V^T$.

Review Problems, XXIV

(Fa20-#12) Consider the matrix $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

4. Find the singular value decomposition $A = U\Sigma V^T$.

- The columns of V are an orthonormal basis of eigenvectors for A^TA: $\mathbf{v}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{5}\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$.
- The columns of U are the values $A\mathbf{v_i}/\sigma_i$, which evaluate to $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus we obtain

$$U = \boxed{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}, \ \Sigma = \boxed{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}}, \ V = \boxed{\begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & -1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}}.$$

Review Problems, XXV

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

1. Solve the linear system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = (x_1, x_2, x_3)$.

Review Problems, XXV

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

1. Solve the linear system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = (x_1, x_2, x_3)$.

- Row-reduce the coefficient matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ -3 & -8 & 7 & -9 \\ 2 & 5 & -3 & 4 \end{array}\right] \underset{R_3-2R_1}{\overset{R_2+3R_1}{\longrightarrow}} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 0 & 1 & -5 & 6 \\ 0 & -1 & 5 & -6 \end{array}\right]$$

$$\overset{R_3-R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1-3R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -13 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- So we have one free variable x_3, with $x_1 = -11x_3 - 13$, $x_2 = 5x_3 + 6$. Then $(x_1, x_2, x_3) = \boxed{(-13, 6, 0) + x_3(-11, 5, 1)}$.

Review Problems, XXVI

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

2. What is $\mathrm{rank}(A)$?

Review Problems, XXVI

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

2. What is $\mathrm{rank}(A)$?
   - This is the number of nonzero rows in the RREF, which is $\boxed{2}$.
3. What is $\mathrm{nullity}(A)$?

Review Problems, XXVI

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

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4. Give a basis for $\ker(A)$.

Review Problems, XXVI

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

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4. Give a basis for $\ker(A)$.
   - From the system solution, we take the basis $\boxed{\{(-11, 5, 1)\}}$.
5. Give a basis for $\mathrm{im}(A)$.

Review Problems, XXVI

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

2. What is $\mathrm{rank}(A)$?
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   - This is the number of zero rows in the RREF, which is $\boxed{1}$.
4. Give a basis for $\ker(A)$.
   - From the system solution, we take the basis $\boxed{\{(-11, 5, 1)\}}$.
5. Give a basis for $\mathrm{im}(A)$.
   - We take the columns of A with pivots in the RREF, which are $\boxed{\{(1, -3, 2), (3, -8, 5)\}}$.
6. What is $\dim(\mathrm{im}\, A)$?

Review Problems, XXVI

(Sp20-#1) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ -3 & -8 & 7 \\ 2 & 5 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$.

2. What is $\mathrm{rank}(A)$?
   - This is the number of nonzero rows in the RREF, which is $\boxed{2}$.
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   - This is the number of zero rows in the RREF, which is $\boxed{1}$.
4. Give a basis for $\ker(A)$.
   - From the system solution, we take the basis $\boxed{\{(-11, 5, 1)\}}$.
5. Give a basis for $\mathrm{im}(A)$.
   - We take the columns of A with pivots in the RREF, which are $\boxed{\{(1, -3, 2), (3, -8, 5)\}}$.
6. What is $\dim(\mathrm{im}\, A)$?
   - This is the rank of A, which is $\boxed{2}$.

Review Problems, XXVII

(Sp20-#2) For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (10x + 3y, 6x + 2y)$, find a formula for the inverse transformation $T^{-1}(x, y)$.

Review Problems, XXVII

(Sp20-#2) For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (10x + 3y, 6x + 2y)$, find a formula for the inverse transformation $T^{-1}(x, y)$.

- This is the linear transformation $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
- The inverse is then $T^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 6 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.
- Using the 2×2 inverse formula we get the matrix $\begin{bmatrix} 10 & 3 \\ 6 & 2 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -6 & 10 \end{bmatrix}$.
- So $T^{-1}(x, y) = \boxed{(x - 3y/2, -3x + 5y)}$.

Review Problems, XXVIII

(Sp20-#3) Given the scaled rotation matrix $R = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and the scaled reflection matrix $F = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, find:

Review Problems, XXVIII

(Sp20-#3) Given the scaled rotation matrix $R = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and the scaled reflection matrix $F = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, find:

1. Find the matrix of the composition $T = R \circ F$.

Review Problems, XXVIII

(Sp20-#3) Given the scaled rotation matrix $R = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and the scaled reflection matrix $F = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, find:

1. Find the matrix of the composition $T = R \circ F$.
   - This is the matrix product $RF = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & -5 \end{bmatrix}$.
2. Find the matrix of the composition $S = F \circ R$.

Review Problems, XXVIII

(Sp20-#3) Given the scaled rotation matrix $R = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and the scaled reflection matrix $F = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, find:

1. Find the matrix of the composition $T = R \circ F$.

- This is the matrix product $RF = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & -5 \end{bmatrix}$.

2. Find the matrix of the composition $S = F \circ R$.

- This is the matrix product $FR = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 7 & 1 \end{bmatrix}$.

Review Problems, XXIX

(Sp20-#4) Define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}$ where $A = \begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix}$ is the matrix of T in the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ for $\mathbb{R}^2$. If the basis for $\mathbb{R}^2$ is changed to $\mathbf{v}_1 = (2, 1)$, $\mathbf{v}_2 = (5, 3)$, what is the matrix representing T in this new basis?

Review Problems, XXIX

(Sp20-#4) Define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}$ where $A = \begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix}$ is the matrix of T in the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ for $\mathbb{R}^2$. If the basis for $\mathbb{R}^2$ is changed to $\mathbf{v}_1 = (2, 1)$, $\mathbf{v}_2 = (5, 3)$, what is the matrix representing T in this new basis?

- If α is the standard basis and β is the new one, the change of basis formula says $[T]_\beta^\beta = P^{-1}[T]_\alpha P$ where $P = [I]_\beta^\alpha = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is the change of basis from β to α.
- Since $P^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, we have
$$[T]_\beta = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}.$$

Review Problems, XXX

(Sp20-#5) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -6 \\ 2 & 8 & -3 \end{bmatrix}$.

1. Calculate A^{-1} by the Gauss-Jordan method.

Review Problems, XXX

(Sp20-#5) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -6 \\ 2 & 8 & -3 \end{bmatrix}$.

1. Calculate A^{-1} by the Gauss-Jordan method.

- Set up the double matrix and row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 4 & -6 & 0 & 1 & 0 \\ 2 & 8 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3-2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 9 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_3/9} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2/9 & 1/9 \end{array}\right] \xrightarrow[R_2+2R_3]{R_1+4R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & -8/9 & 4/9 \\ 0 & 1 & 0 & -1 & 5/9 & 2/9 \\ 0 & 0 & 1 & 0 & -2/9 & 1/9 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -23/9 & -2/9 \\ 0 & 1 & 0 & -1 & 5/9 & 2/9 \\ 0 & 0 & 1 & 0 & -2/9 & 1/9 \end{array}\right]$$

2. Find $\mathrm{rank}(A)$ and $\det(A)$.

Review Problems, XXX

(Sp20-#5) Let $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -6 \\ 2 & 8 & -3 \end{bmatrix}$.

1. Calculate A^{-1} by the Gauss-Jordan method.
   - Set up the double matrix and row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 4 & -6 & 0 & 1 & 0 \\ 2 & 8 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3-2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 9 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_3/9} \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2/9 & 1/9 \end{array}\right] \xrightarrow[R_2+2R_3]{R_1+4R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & -8/9 & 4/9 \\ 0 & 1 & 0 & -1 & 5/9 & 2/9 \\ 0 & 0 & 1 & 0 & -2/9 & 1/9 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -23/9 & -2/9 \\ 0 & 1 & 0 & -1 & 5/9 & 2/9 \\ 0 & 0 & 1 & 0 & -2/9 & 1/9 \end{array}\right]$$

2. Find $\text{rank}(A)$ and $\det(A)$.
   - Since A is invertible its rank is $\boxed{3}$, there were no swaps, and the only rescaling was a division by 9, so $\det(A) = 9$.

Review Problems, XXXI

(Sp20-#6) Prove that the vectors $\mathbf{v}_1 = (1, -3, 4)$, $\mathbf{v}_2 = (-2, 7, 6)$, $\mathbf{v}_3 = (7, -23, 0)$ are linearly dependent, expressing one of them as a linear combination of the others.

Review Problems, XXXI

(Sp20-#6) Prove that the vectors $\mathbf{v}_1 = (1, -3, 4)$, $\mathbf{v}_2 = (-2, 7, 6)$, $\mathbf{v}_3 = (7, -23, 0)$ are linearly dependent, expressing one of them as a linear combination of the others.

- We want scalars a_1, a_2, a_3 with $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (0, 0, 0)$, which is equivalent to the matrix system
$$\begin{bmatrix} 1 & -2 & 7 \\ -3 & 7 & -23 \\ 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
- Row-reducing the coefficient matrix yields the equivalent $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so we have a nontrivial solution $(a_1, a_2, a_3) = (-3, 2, 1)$.
- Thus, $\boxed{-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}}$.

Review Problems, XXXI

(Sp20-#7) Given that the vectors $\mathbf{v}_1 = (-1, 3, -4)$, $\mathbf{v}_2 = (3, -8, 10)$, $\mathbf{v}_3 = (2, -9, 7)$ are linearly independent (hence form a basis for $\mathbb{R}^3$, express $\mathbf{w} = (1, -14, 5)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Review Problems, XXXI

(Sp20-#7) Given that the vectors $\mathbf{v}_1 = (-1, 3, -4)$, $\mathbf{v}_2 = (3, -8, 10)$, $\mathbf{v}_3 = (2, -9, 7)$ are linearly independent (hence form a basis for $\mathbb{R}^3$, express $\mathbf{w} = (1, -14, 5)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

- We want scalars a_1, a_2, a_3 with $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (1, -14, 5)$, which is equivalent to the matrix system $\begin{bmatrix} -1 & 3 & 2 \\ 3 & -8 & -9 \\ -4 & 10 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \\ 5 \end{bmatrix}$.
- Row-reducing the augmented coefficient matrix yields $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ so we get $(a_1, a_2, a_3) = (-1, -2, 3)$.

  Thus, $\mathbf{w} = \boxed{-\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3}$.

Review Problems, XXXIII

(Sp20-#8) Let W be the subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $\mathbf{v}_1 = (-1, 2, 2)$ and $\mathbf{v}_2 = (3, -3, 0)$.

1. Use Gram-Schmidt to find an orthonormal basis for W.

Review Problems, XXXIII

(Sp20-#8) Let W be the subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $\mathbf{v}_1 = (-1, 2, 2)$ and $\mathbf{v}_2 = (3, -3, 0)$.

1. Use Gram-Schmidt to find an orthonormal basis for W.

- First, we take $\mathbf{w}_1 = \mathbf{v}_1 = (-1, 2, 2)$.
- Next, $\mathbf{w}_2 = \mathbf{v_2} - a_1\mathbf{w}_1$ where $a_1 = \dfrac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{-9}{9} = -1$, so $\mathbf{w}_2 = (-1, 2, 2) + 1(3, -3, 0) = (2, -1, 2)$.
- Finally, we normalize to get $\boxed{(-1, 2, 2)/3,\ (2, -1, 2)/3}$.

Review Problems, XXXIV

(Sp20-#8) Let W be the subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $\mathbf{v}_1 = (-1, 2, 2)$ and $\mathbf{v}_2 = (3, -3, 0)$.

2. Find the matrix M of the orthogonal projection $P : \mathbb{R}^3 \to W$.

Review Problems, XXXIV

(Sp20-#8) Let W be the subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $\mathbf{v}_1 = (-1, 2, 2)$ and $\mathbf{v}_2 = (3, -3, 0)$.

2. Find the matrix M of the orthogonal projection $P : \mathbb{R}^3 \to W$.

- We could just compute $M = A(A^T A)^{-1} A^T$ for A with columns $\mathbf{v}_1, \mathbf{v}_2$. But it is quicker to use the orthonormal basis, with $A = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}$, since then $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Either way, $M = AA^T = \boxed{\frac{1}{9}\begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}}$.

3. What is the rank of M?

Review Problems, XXXIV

(Sp20-#8) Let W be the subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $\mathbf{v}_1 = (-1, 2, 2)$ and $\mathbf{v}_2 = (3, -3, 0)$.

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- We could just compute $M = A(A^T A)^{-1} A^T$ for A with columns $\mathbf{v}_1, \mathbf{v}_2$. But it is quicker to use the orthonormal basis, with $A = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}$, since then $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Either way, $M = AA^T = \boxed{\frac{1}{9}\begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}}$.

3. What is the rank of M?

- The column space (i.e., image) of M is W by definition, so the rank is $\dim(W) = \boxed{2}$.

Review Problems, XXXIV

(Sp20-#9) Evaluate the determinant $\begin{vmatrix} 2 & 1 & -4 & 0 \\ 7 & 3 & -13 & 8 \\ -5 & -2 & 11 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix}$.

Review Problems, XXXIV

(Sp20-#9) Evaluate the determinant $\begin{vmatrix} 2 & 1 & -4 & 0 \\ 7 & 3 & -13 & 8 \\ -5 & -2 & 11 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix}$.

- Using row operations and expansion along rows/columns we get $\begin{vmatrix} 2 & 1 & -4 & 0 \\ 7 & 3 & -13 & 8 \\ -5 & -2 & 11 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix} \underset{R_3+2R_1}{\overset{R_2-3R_1}{=}} \begin{vmatrix} 2 & 1 & -4 & 0 \\ 1 & 0 & -1 & 8 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{vmatrix} \overset{R_4}{=}$ $6 \cdot \begin{vmatrix} 2 & 1 & -4 \\ 1 & 0 & -1 \\ -1 & 0 & 3 \end{vmatrix} \overset{C_2}{=} 6 \cdot (-1) \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 6 \cdot (-1) \cdot 2 = \boxed{-12}$.

Review Problems, XXXIV

(Sp20-#10) Given that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 4 & -4 & 10 \\ 0 & 0 & 2 \end{bmatrix}$, find an associated eigenvector corresponding to $\lambda = 2$.

Review Problems, XXXIV

(Sp20-#10) Given that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 4 & -4 & 10 \\ 0 & 0 & 2 \end{bmatrix}$, find an associated eigenvector corresponding to $\lambda = 2$.

- The 2-eigenspace is the nullspace (kernel) of $2I_3 - A = \begin{bmatrix} -1 & 1 & -1 \\ -4 & 6 & -10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}$.
- So the eigenspace is 1-dimensional.
- Solving the system $-a + b - c = 0, 2b - 6c = 0$ yields $b = 3c, a = 2c$ so we have the basis $\boxed{\{(2, 3, 1)\}}$.

Review Problems, XXXV

(Sp20-#11) Given the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, find an orthonormal basis diagonalizing A, and the diagonalization D.

Review Problems, XXXV

(Sp20-#11) Given the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, find an orthonormal basis diagonalizing A, and the diagonalization D.

- We just need to find an orthonormal basis of eigenvectors.
- The characteristic polynomial is $\det(tI_2 - A) = \begin{vmatrix} t-1 & -2 \\ -2 & t-1 \end{vmatrix} = (t-3)(t+1)$, so $\lambda = -1, 3$.
- The (-1)-eigenspace is the nullspace of $-I_2 - A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ giving $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- The 3-eigenspace is the nullspace of $3I_2 - A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ giving $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Basis $\boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ with $D = \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}}$.

Summary

We did some review problems for the final exam.

(Mostly for my students) Please fill out the TRACE evaluations!

(Mostly for not-my students) In the spring semester I am teaching Math 4571 (Advanced Linear Algebra) which is essentially Math 2331 but from a theoretical standpoint (i.e., where we prove everything, and problems are mostly conceptual). If you've taken 1365 and are a math or physics or CS major, I highly recommend looking into 4571 – linear algebra is incredibly useful stuff.

(For all students) Good luck on the final exam!