E. Dummit's Math 5111 ~ Algebra 1, Fall 2020 ~ Homework 9, due Fri Nov 20th.

Justify all responses with proof and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. You may use results from earlier parts of problems in later parts, even if you were unable to solve the earlier parts.

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1. The goal of this problem is to classify the groups of order 8 up to isomorphism.
   (a) If G is a non-abelian group of order 8, show that G has an element a of order 4 that generates a normal subgroup N of G. [Hint: Use problem 2 of homework 7 and problem 1 of homework 8.]
   (b) Continuing (b), if bN is the other left coset of N, show that bab^{-1} is equal to a^{-1}.
   (c) Continuing (c), if b has order 2 show that G is isomorphic to $D_{2\cdot 4}$, and if b has order 4 show that G is isomorphic to Q_8.
   (d) Classify the groups of order 8 up to isomorphism.

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2. Classify the abelian groups of order 32, 7700, and 1800 up to isomorphism in both invariant factor form and elementary divisor form. For convenience, you may abbreviate the cyclic group $\mathbb{Z}/n\mathbb{Z}$ as C_n.

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3. Prove or disprove: If G is a group with subgroups H_1 and H_2 such that H_1 is isomorphic to H_2, then G/H_1 is isomorphic to G/H_2.

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4. Solve the following problems:
   (a) Suppose G is a group of order 2020. Find all possibilities for the Sylow numbers of G (per the conditions of Sylow's theorems), and prove that G cannot be simple.
   (b) Suppose G is a group of order 56. Find all possibilities for the Sylow numbers of G (per the conditions of Sylow's theorems), and prove that G cannot be simple.
   (c) Suppose G is a group of order 2021. Prove that G is cyclic.
   (d) Suppose G is a group of order 11011. Prove that G is abelian.

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5. Let p be a prime and let $G = SL_2(\mathbb{F}_p)$.
   (a) Find the number of Sylow p-subgroups of G. [Hint: The upper and lower triangular matrices with 1s on the diagonal give two different Sylow p-subgroups.]
   (b) Find the number of elements in G of order p.

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6. If G is an additive abelian group (not necessarily finitely generated), we say a subset S of G is <u>linearly independent</u> if for any distinct $g_1, g_2, \dots, g_n \in S$, the only integers $a_1, a_2, \dots, a_n$ such that $a_1 g_1 + a_2 g_2 + \cdots + a_n g_n = 0$ are $a_1 = a_2 = \cdots = a_n = 0$.
   (a) If S is a linearly independent set, show that every element of S has infinite order.
   (b) Show that G must possess a maximal linearly independent set.
   (c) We define the (free) <u>rank</u> of an additive abelian group G to be the cardinality of any maximal linearly independent set in G. By a replacement argument, one may show that the free rank is well-defined. If G is finitely generated and $G \cong \mathbb{Z}^r \times (\mathbb{Z}/a_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/a_k\mathbb{Z})$ is in invariant factor form, show that the free rank of G is r.

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7. The goal of this problem is to illustrate some applications of group theory in enumerative combinatorics.

   (a) Let G be a finite group acting on a finite set A. For each $g \in G$, let $A^g = \{a \in A : g \cdot a = a\}$ denote the set of elements in A that are fixed by g. Prove the Cauchy-Frobenius lemma: the number of orbits of G on A is equal to $\frac{1}{\#G} \sum_{g \in G} \#A^g$. [Hint: Explain why $\sum_{g \in G} \#A^g = \sum_{a \in A} \#G_a$, where G_a is the stabilizer of a, and use the orbit-stabilizer theorem.]

   (b) A total of 2020 fine gentlemen attend a restaurant in the 1920s, and they of course all check their hats. However, the hat-checker forgets to keep track of whose hats are whose, and thus returns them in random order. What is the expected number of fine gentlemen who receive their own hat back?

   (c) Let G be a finite group acting on a finite set X of beads. Each of the beads is colored in one of m colors, and we view two colorings of X as equivalent if and only if there exists an element of G mapping one coloring to the other. Prove Pólya's enumeration formula: the total number of inequivalent colorings is equal to $\frac{1}{\#G} \sum_{g \in G} m^{c(g)}$, where $c(g)$ is the number of cycles of the element g when considered as a permutation of X.

   (d) The four faces of a tetrahedron are each painted one of the six colors periwinkle, peridot, burgundy, chartreuse, cerulean, and eggshell. Find the total number of different colorings up to rotations. [Hint: Note that the group of rigid symmetries of the tetrahedron is isomorphic to A_4.]

   (e) Ten beads are arranged around a circular necklace. Each bead is colored vermillion, cornflower, or fuchsia. Necklaces are equivalent if they can be rotated into one another. Find the number of inequivalent necklaces.

   (f) Let p be a prime and a be a positive integer. Consider circular necklaces with p equally-spaced beads, each of which can be colored in one of a colors. Find the total number of distinct necklaces, up to rotation. Deduce that $a^p \equiv a \pmod{p}$ for all primes p and positive integers a.
