E. Dummit's Math 5111 ~ Algebra 1, Fall 2020 ~ Homework 8, due Fri Nov 6th.

Justify all responses with proof and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. You may use results from earlier parts of problems in later parts, even if you were unable to solve the earlier parts.

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1. Let G be a group.

   (a) Suppose H is a subgroup of G and $[G : H] = 2$. Prove that H is normal in G. [Hint: Show the left and right cosets of H are the same.]

   (b) Suppose H is a finite subgroup of G, and there is no other subgroup of G having the same order as H. Prove that H is normal in G. [Hint: Conjugate H.]

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2. Let G be a group and $Z(G) = \{a \in G : ga = ag \text{ for all } g \in G\}$ be the center of G. Recall that $Z(G)$ is a normal subgroup of G.

   (a) Suppose that $G/Z(G)$ is cyclic. Show that there exists an element $g \in G$ such that every element of G can be written in the form $g^k z$ for some integer k and some $z \in Z(G)$.

   (b) Suppose that $G/Z(G)$ is cyclic. Show that G is abelian.

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3. Suppose G is the group with presentation $\langle x, y : x^7 = y^6 = e,\ yx = x^5 y\rangle$.

   (a) Show that every element of G can be written as $x^a y^b$ for some integers a and b. Conclude that $|G| \leq 42$.

   (b) Show that there exists a homomorphism $\varphi : G \to S_7$ such that $\varphi(x) = (1\,2\,3\,4\,5\,6\,7)$ and $\varphi(y) = (2\,6\,5\,7\,3\,4)$.

   (c) Prove that $|G| = 42$. [Hint: Use Lagrange's theorem to obtain a lower bound on the order of $\text{im}(\varphi)$ from part (b).]

   - Remark: This group is called the Frobenius group of order 42.

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4. Find the following things:

   (a) If $n \geq 2$, find the orbits of S_n acting on ordered pairs of elements of $\{1, 2, \ldots, n\}$ via $\sigma \cdot (a, b) = (\sigma(a), \sigma(b))$ for each $\sigma \in S_n$.

   (b) Find the orbit and the stabilizer of $x_1 + x_2$ and of $x_1 x_2 x_3$ under the action of S_4 on polynomials $F[x_1, x_2, x_3, x_4]$ by index permutation.

   (c) Suppose we label the elements of $G = D_{2\cdot 4} = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ in the order $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (so that s corresponds to 5, etc.), and we let G act on itself by left multiplication. Identify the permutations in S_8 corresponding to left multiplication by the elements r and s.

   (d) Find an explicit permutation $\tau \in S_7$ such that $\tau(1\,2\,4)(5\,6)\tau^{-1} = (2\,5)(3\,4\,7)$.

   (e) Find an explicit permutation $\tau \in S_8$ such that $\tau\sigma\tau^{-1} = \sigma^2$, where $\sigma = (1\,2\,3\,4\,5)(6\,7\,8)$.

   (f) Find the conjugacy classes in $D_{2\cdot 4}$ and in Q_8.

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5. Let A be the group of rigid motions of a regular tetrahedron in 3-space. (Rigid motions are symmetries that involve only translation and rotation, not reflection.)

   (a) Show that $|A| = 12$ and that A is isomorphic to a subgroup of S_4.

   (b) Show that A is isomorphic to A_4. [Hint: Count elements of order 3.]

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6. If G is a group, an isomorphism $\varphi : G \to G$ is called a (group) <u>automorphism</u> of G. Intuitively, automorphisms represent structure-preserving symmetries of G.

   (a) If $G = \mathbb{Z}/n\mathbb{Z}$ and a is a unit modulo n, show that the map $\varphi_a : G \to G$ given by $\varphi_a(x) = ax$ is an automorphism of G.

   (b) Show that the set of all automorphisms of G, denoted $\text{Aut}(G)$, is itself a group under function composition.

   (c) Show that the automorphism group of $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$. [Hint: Show the maps from (a) are the only automorphisms.]

   For any $g \in G$, define the conjugation-by-g map $\sigma_g : G \to G$ via $\sigma(x) = gxg^{-1}$ for any $x \in G$. As shown in class, σ_g is an automorphism of G.

   (d) Show that $\sigma_{gh} = \sigma_g\sigma_h$ and $\sigma_g^{-1} = \sigma_{g^{-1}}$. Deduce that the set of automorphisms of the form σ_g for some $g \in G$ is a subgroup of $\text{Aut}(G)$. This set is denoted $\text{Inn}(G)$, the collection of <u>inner automorphisms</u> of G.

   (e) If $\varphi : G \to G$ is any automorphism of G, show that $\varphi \circ \sigma_g \circ \varphi^{-1} = \sigma_{\varphi(g)}$. Deduce that $\text{Inn}(G)$ is in fact a normal subgroup of G. [Hint: It is perhaps easier to verify that $\varphi \circ \sigma_g = \sigma_{\varphi(g)} \circ \varphi$.]

   (f) Show that any group of order > 2 has a nontrivial automorphism.

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7. If I is any indexing set and $(G_i, \cdot_i)$ is a group for each $i \in I$, the <u>direct product</u> of the groups G_i has underlying set $G = \prod_{i \in I} G_i$, the Cartesian product of the corresponding sets, and group operation defined componentwise: $(\prod_{i \in I} a_i) \cdot (\prod_{i \in I} b_i) = (\prod_{i \in I} a_i \cdot_i b_i)$. It is straightforward to verify that the direct product is a group.

   (a) Show that the group $\prod_{d=1}^{\infty}(\mathbb{Z}/d\mathbb{Z})$ has an element of order n for each positive integer n, and also has elements of infinite order.

   (b) Show that there exists an infinite group in which every element has order dividing 2.

   (c) Show that there exists an infinite group that is isomorphic to a proper subgroup of itself.

   (d) Show that there exists an infinite group G such that G is isomorphic to $G \times G$.

   (e) Sharpen part (c) by showing that there exist infinite groups G and H such that G is isomorphic to a subgroup of H, and H is isomorphic to a subgroup of G, but G and H are not isomorphic. (This yields a counterexample to the group analogue of the Cantor-Schröder-Bernstein theorem.)

   (f) The <u>direct sum</u> $\bigoplus_{i \in I} G_i$ of the groups G_i is the set of elements in the direct product having only finitely many components not equal to e. Show that the direct sum is a normal subgroup of the direct product.

   (g) Show that the direct sum $\bigoplus_{d=1}^{\infty}(\mathbb{Z}/d\mathbb{Z})$ has an element of order n for each positive integer n, but has no elements of infinite order.
