E. Dummit's Math 5111 ~ Algebra 1, Fall 2020 ~ Homework 4, due Fri Oct 9th.

Justify all responses with proof and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. You may use results from earlier parts of problems in later parts, even if you were unable to solve the earlier parts.

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1. Show that the numbers $\alpha = \sqrt{2} + \sqrt[3]{2}$, $\beta = \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{10} + \sqrt{11} + \sqrt{13}$, and $\gamma = 1 + 2\sqrt[3]{4 + 5\sqrt[6]{7} + \sqrt[8]{9}}$ are algebraic over $\mathbb{Q}$, and find explicit upper bounds on the degrees of their minimal polynomials over $\mathbb{Q}$.

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2. Suppose that $p_1, p_2, \dots, p_n$ are integers.

   (a) Prove that $[\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}) : \mathbb{Q}]$ divides 2^n.

   (b) Prove that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13})$.

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3. Determine the degree of $L = \mathbb{Q}(2^{1/2}, 2^{1/5})$ over $\mathbb{Q}$.

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4. Let $a, b \in \mathbb{Q}$ such that $\sqrt{b} \notin \mathbb{Q}$, and define $\alpha = \sqrt{a + \sqrt{b}} \in \mathbb{C}$.

   (a) Show that there exist $m, n \in \mathbb{Q}$ such that $\alpha = \sqrt{m} + \sqrt{n}$ if and only if $a^2 - b$ is the square of a rational number.

   (b) If $\beta = \sqrt{2 + \sqrt{3}}$ and $\gamma = \sqrt{2 - \sqrt{3}}$, find $[\mathbb{Q}(\beta) : \mathbb{Q}]$, $[\mathbb{Q}(\gamma) : \mathbb{Q}]$, and $[\mathbb{Q}(\beta + \gamma) : \mathbb{Q}]$.

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5. Suppose K/F is a finite-degree field extension and let $\beta = \{\beta_1, \beta_2, \dots, \beta_n\}$ be a basis of K/F. If $T : K \to K$ is a linear transformation, we define the associated matrix $[T]_\beta^\beta \in M_{n \times n}(F)$ to be the matrix whose (i, j)-entry is the coefficient of β_j when $T(\beta_i)$ is written as a linear combination of the elements of the basis β. In other words, if $T_\alpha(\beta_i) = \alpha\beta_i = c_{1,i}\beta_1 + c_{2,i}\beta_2 + \cdots + c_{n,i}\beta_n$ then the ith column of the matrix M_α is the column vector $\langle c_{1,i}, c_{2,i}, \dots, c_{n,i} \rangle^T$.

   (a) [Optional] Prove that the association Φ of a linear transformation with its associated matrix is a ring isomorphism between the space $\mathcal{L}$ of linear transformations from K to K (with operations of pointwise addition and composition) and the matrix ring $M_{n \times n}(F)$.

   Now, for any $\alpha \in K$, define the linear transformation $T_\alpha : K \to K$ via $T_\alpha(v) = \alpha v$ and let M_α be its associated matrix in the basis β.

   (b) For $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt[3]{2})$ with basis $\beta = \{1, \sqrt[3]{2}, \sqrt[3]{4}\}$, verify that $M_{\sqrt[3]{2}} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and determine $M_{1+\sqrt[3]{2}+\sqrt[3]{4}}$.

   (c) If $\alpha \neq 0$, show that T_α is in fact a vector space isomorphism of K with itself, and that its inverse map is given by $T_\alpha^{-1} = T_{\alpha^{-1}}$.

   (d) Part (c) says that we can compute the multiplicative inverse of any nonzero $\alpha \in K$ by computing the inverse of the corresponding matrix M_α: specifically, the coefficients of α^{-1} in the basis β will be the first column of the matrix M_α^{-1}. Use this procedure to compute the multiplicative inverse of $\sqrt[3]{2}$ and of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ in $\mathbb{Q}(\sqrt[3]{2})$.

(e) Show that α is an eigenvalue[1] of M_α. Deduce that α is a root of the characteristic polynomial[2] of M_α. [Hint: Show that the column vector $w = \langle \beta_1, \beta_2, \ldots, \beta_n \rangle^T$ is an eigenvector of the transpose of M_α over K.]

(f) Part (e) says that α is a root of the polynomial $p(t) = \det(tI - M_\alpha) \in F[t]$. Use this result to find a nonzero polynomial in $\mathbb{Q}[t]$ having $\sqrt[3]{2}$ as a root, and also one having $1 + \sqrt[3]{2} + \sqrt[3]{4}$ as a root.

(g) If $K = F(\alpha)$, prove that the characteristic polynomial $p(t) = \det(tI - M_\alpha)$ of M_α is the minimal polynomial of α. Use this result to find the minimal polynomial of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.

(h) Prove that every finite-degree field extension embeds into (i.e., is isomorphic to a subring of) a matrix ring. [Hint: Consider the collection of all matrices in $M_{n\times n}(F)$ of the form M_α for $\alpha \in K$.]

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[1] Recall that an eigenvector of a linear transformation is a nonzero vector v with $T(v) = \lambda v$ for some $\lambda \in K$, and λ is the associated eigenvalue.

[2] Recall that the characteristic polynomial of a matrix A is the polynomial $p(t) = \det(tI - A)$, and that the eigenvalues of A are precisely the roots of $p(t)$. The characteristic polynomial of the transpose of a matrix is the same as the characteristic polynomial of the matrix.