E. Dummit's Math 5111 ~ Algebra 1, Fall 2020 ~ Homework 3, due Fri Oct 2nd.

Justify all responses with proof and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. You may use results from earlier parts of problems in later parts, even if you were unable to solve the earlier parts.

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1. Let R be a ring with 1. Prove that R contains a maximal ideal (i.e., an ideal $M \neq R$ that is maximal under containment). [Hint: Use Zorn's lemma.]

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2. For each α over each field F, show that α is algebraic over F by finding its minimal polynomial $m(x) \in F[x]$:
   (a) $\alpha = 4 + 3i$ over $F = \mathbb{Q}$.
   (b) $\alpha = 4 + 3i$ over $F = \mathbb{Q}(\sqrt{3})$.
   (c) $\alpha = 2 + \sqrt[3]{2}$ over $F = \mathbb{Q}$.
   (d) $\alpha = \sqrt{2} + \sqrt{5}$ over $F = \mathbb{Q}$.
   (e) $\alpha = \sqrt{2} + \sqrt{5}$ over $F = \mathbb{Q}(\sqrt{2})$.

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3. Let $K = \mathbb{Q}(3^{1/12})$.
   (a) Find a basis and the relative degree for $K/\mathbb{Q}$.
   (b) Find a basis and the relative degree for $K/\mathbb{Q}(\sqrt{3})$.

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4. Suppose K/F is a field extension. If $a, b \in F$ with $a \neq 0$, show that $F(r) = F(ar + b)$ for any $r \in K$.

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5. Suppose $K = F(\alpha)$ where $\alpha \in K$ is nonzero and algebraic over F.
   (a) Show that α is the root of some polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ in $F[x]$ with $a_0 \neq 0$.
   (b) With notation as in part (a), show that $\alpha^{-1} = -\frac{1}{a_0}\left[a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1}\right]$.
   (c) Find a polynomial $p(x) \in \mathbb{Q}[x]$ of which $\sqrt{3}+\sqrt{5}$ is a root, and then use (b) to compute the multiplicative inverse of $\sqrt{3} + \sqrt{5}$ in $\mathbb{Q}(\sqrt{3} + \sqrt{5})$.

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6. Suppose $K = F(\alpha)$ where α is algebraic over F with minimal polynomial $m(x) \in F[x]$. Suppose $\beta = p(\alpha)$ is a nonzero element of K, where $p(x) \in F[x]$.
   (a) Show that there exists a polynomial $q(x) \in F[x]$ such that $m(x)$ divides $p(x)q(x) - 1$. Conclude that $p(\alpha)^{-1} = q(\alpha)$ in K. [Hint: Euclid.]
   (b) Use the procedure of part (a) to compute the multiplicative inverse of $1 + \sqrt[3]{9}$ in $\mathbb{Q}(\sqrt[3]{3})$.

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7. Recall that π is transcendental over $\mathbb{Q}$.
   (a) Show that $\mathbb{Q}(\pi)$ is isomorphic (as a field) to $\mathbb{Q}(\pi^2)$.
   (b) Show that π is algebraic over $\mathbb{Q}(\pi^6)$ and over $\mathbb{Q}(\pi^2 + \pi)$.
   (c) Show that $\mathbb{Q}(\pi)$ is a field extension of $\mathbb{Q}(\pi^2)$ of degree 2.
   - Remark: To emphasize, this problem gives an example of fields E and F such that E is a proper subfield of F, even though E and F are isomorphic as fields.

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8. The goal of this problem is to generalize the ideas from problem 7. Let F be a field and $F(x)$ be the field of rational functions in the indeterminate x. Let $p, q \in F[x]$ be relatively prime with $q \neq 0$, and define $t = \dfrac{p(x)}{q(x)} \in F(x)$. The goal of this problem is to prove that $F(x)/F(t)$ is algebraic and that $[F(x) : F(t)] = \max(\deg p, \deg q)$.

   (a) Suppose $f \in F[x, y]$ is a polynomial in the two variables x and y. Prove the following version of Gauss's lemma: if p is irreducible as an element of the polynomial ring $[F(x)][y]$ in the variable y with coefficients in $F(x)$, then it is also irreducible as an element of the polynomial ring $[F[x]][y] = F[x, y]$ in the variable y with coefficients in $F[x]$. [Hint: Use the same argument as in class, with $F[x]$ and $F(x)$ in place of $\mathbb{Z}$ and $\mathbb{Q}$.]

   (b) Prove that the polynomial $g(y) = p(y) - t\,q(y)$ in the variable y and coefficients in $F(t)$ is irreducible over $F(t)$ and has x as a root. [Hint: Use (a) along with $[F[t]][y] \cong [F[y]][t]$ for the irreducibility.]

   (c) Show that $[F(x) : F(t)]$ is algebraic and that its degree is $\max(\deg p, \deg q)$. [Hint: Why is $g(y) = p(y) - t\,q(y)$ is the minimal polynomial of x over $F(t)$, and what is its degree?]