E. Dummit's Math 5111 ~ Algebra 1, Fall 2020 ~ Homework 10, due Fri Nov 27th.

Justify all responses with proof and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. You may use results from earlier parts of problems in later parts, even if you were unable to solve the earlier parts.

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1. Find the number of automorphisms for each of the given field extensions, and identify the corresponding automorphism group structure:
   (a) $\mathbb{Q}(\sqrt[5]{2})/\mathbb{Q}$.
   (b) $\mathbb{C}/\mathbb{R}$.
   (c) $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$.
   (d) $\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q}$.
   (e) K/F, where $F = \mathbb{F}_5(t)$ and K is the splitting field of $p(x) = x^5 - t$ over F. [Hint: p is inseparable.]

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2. For each algebraic number α, determine all of its Galois conjugates over $\mathbb{Q}$, the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, and the minimal polynomial $m(x)$ of α over $\mathbb{Q}$ (it does not need to be simplified or expanded):
   (a) $\alpha = \sqrt{2} + 4\sqrt{3}$.
   (b) $\alpha = \sqrt{2} + \sqrt{3} + \sqrt{6}$.
   (c) $\alpha = 1 + 3 \cdot 2^{1/3} + 4^{1/3}$.
   (d) $\alpha = 3^{1/4}i$.
   (e) $\alpha = 3^{1/4} + i$.
   (f) $\alpha = \zeta_8 = e^{2\pi i/8}$.

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3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Note that $[K : \mathbb{Q}] = 8$ (you may use this fact freely in this problem).
   (a) Show that K is Galois over $\mathbb{Q}$ and that the Galois group of K is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.
   (b) Find all of the subfields E of K with $[E : \mathbb{Q}] = 4$. [Hint: There are 7 of them.]
   (c) Find all of the subfields E of K with $[E : \mathbb{Q}] = 2$. [Hint: There are 7 of them.]

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4. Let p be a prime and $a \in \mathbb{F}_p$ be nonzero. Show that the Galois group of $q(x) = x^p - x + a$ over $\mathbb{F}_p$ is cyclic of order p. [Hint: One way uses problem 1(d) from homework 6, while another computes the Frobenius map.]

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5. Let $K = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{2 + \sqrt{2}}$.
   (a) Show that $[K : \mathbb{Q}] = 4$.
   (b) Show that K is a Galois extension of $\mathbb{Q}$.
   (c) Show that $\text{Gal}(K/\mathbb{Q})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. [Hint: Show that the map with $\sigma(\alpha) = \sqrt{2}/\alpha$ extends to an automorphism of order 4. You may want to compute $\sigma(\sqrt{2})$ first.]

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6. Let F be a field. The goal of this problem is to determine $\mathrm{Aut}(F(t)/F)$, where t represents an indeterminate. Recall that if $y = p(t)/q(t)$ where $p(t)$ and $q(t)$ are relatively prime, then the extension degree $[F(t) : F(y)] = \max(\deg p, \deg q)$.

   (a) Suppose that σ is an automorphism of $F(t)$. Show that $\sigma(t) = \dfrac{at+b}{ct+d}$ for some $a, b, c, d \in F$ with $ad - bc \neq 0$. [Hint: Use the extension degree formula.]

   (b) Suppose that $a, b, c, d \in F$ and that $ad - bc \neq 0$. Show that the map σ sending $p(t) \in F(t)$ to $p(\dfrac{at+b}{ct+d})$ is an automorphism of $F(t)$.

   (c) Conclude that the automorphisms of $F(t)/F$ are precisely the maps described in part (b).

   (d) Prove that there is a group homomorphism $\varphi : GL_2(F) \to \mathrm{Aut}(F(t)/F)$ such that $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$ is the automorphism sending $t \mapsto \dfrac{at+b}{ct+d}$.

   (e) Deduce that $\mathrm{Aut}(F(t)/F)$ is isomorphic to $GL_2(F)/N$, where N is the normal subgroup consisting of nonzero scalar multiples of the identity matrix. This quotient group is called the <u>projective general linear group</u> on F^n and denoted $PGL_2(F)$.

   - <u>Remark</u>: These automorphisms are called <u>fractional linear transformations</u>, and play an important role in many areas of mathematics, including algebra, geometry, and complex analysis (when $F = \mathbb{C}$ they are called Möbius transformations).

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7. Let $K = \mathbb{C}(t)$ and consider the automorphisms σ and τ of $K/\mathbb{C}$ such that $\sigma(t) = \zeta_n t$ and $\tau(t) = t^{-1}$. Note that both of these choices do extend to automorphisms by problem 6.

   (a) Show that the subgroup of $\mathrm{Aut}(K/\mathbb{C})$ generated by σ and τ has order $2n$ and is isomorphic to the dihedral group $D_{2 \cdot n}$.

   (b) Show that the fixed field of $\langle \sigma, \tau \rangle$ is $\mathbb{C}(t^n + t^{-n})$. [Hint: Show this field is fixed by σ and τ, and then use the extension degree formula.]

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8. The goal of this problem is to determine $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$. Suppose $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

   (a) Show that σ sends positive real numbers to positive real numbers. [Hint: Positive real numbers are squares.]

   (b) Show that σ is a strictly increasing function (i.e., if $a < b$ then $\sigma(a) < \sigma(b)$).

   (c) Show that if m is a positive integer and $-\dfrac{1}{m} < b - a < \dfrac{1}{m}$, then $-\dfrac{1}{m} < \sigma(b) - \sigma(a) < \dfrac{1}{m}$. Deduce that σ is continuous.

   (d) Show that a continuous function on $\mathbb{R}$ that fixes $\mathbb{Q}$ must be the identity map, and conclude that $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is the trivial group.
