Calculus III (part 4): Vector Calculus (by Evan Dummit, 2020, v. 3.50)

Contents

4 Vector Calculus

Our motivating problem for multivariable integration was to generalize the idea of integration to more complicated regions in space, or (more succinctly) to integrate a function over a region. We might also ask whether there is a simple way to integrate a function over an arbitrary curve in the plane or in space, and whether there is a way to integrate a function over an arbitrary surface in space. The answer (as it always has been to this point) is yes: the generalization of single-variable integration to arbitrary curves is called a line integral, and the generalization of double integration to arbitrary surfaces is called a surface integral.

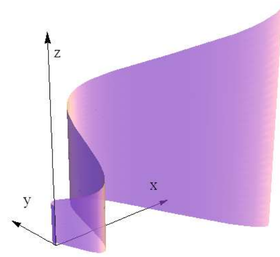
After introducing line and surface integrals, we will then discuss vector fields (which are vector-valued functions in 2-space and 3-space) which provide a useful model for the flow of a fluid through space. The principal applications of line and surface integrals are to the calculation of the work done by a vector field on a particle traveling through space, the flux of a vector field across a curve or through a surface, and the circulation of a vector field along a curve.

Finally, we discuss several generalizations of the Fundamental Theorem of Calculus: the Fundamental Theorem of Calculus for line integrals, Green's Theorem, Gauss's Divergence Theorem, and Stokes's Theorem. Collectively, these theorems unify all of the different notions of integration, as they each relate the integral of a function on a region to the integral of an antiderivative of the function on the region's boundary.

4.1 Line Integrals

- The motivating problem for our discussion of line integrals is: given a parametric curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and a function $f(x, y)$, if we "build a surface" along the curve with height given by the function $z = f(x, y)$, how can we calculate the area of this surface? (This is a natural generalization of our typical single-variable integration problem, in which we build the "surface" inside a plane, thus making it the area under a curve.)

- Here is an example (for visualization), with $\mathbf{r}(t) = \langle t^2,\, t\cos(2\pi t)\rangle$, $f(x, y) = t^2 + 1$, for $0 \le t \le \dfrac{3}{2}$:



- Another closely related question is: given a parametric curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and a function $f(x, y, z)$, how can we calculate the average value of $f(x, y, z)$ on the curve?
- A third question: given a thin wire shaped along some curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with variable density $\delta(x, y)$, what is the wire's mass, and what are its moments about the coordinate axes?

- As with all other types of integrals we have examined so far, we use Riemann sums to give the formal definition of the line integral of a function $f(x, y)$ on a plane curve C. (Also as before, we will use the formal definition as infrequently as possible!)
  - The idea is to approximate the curve with straight line segments, sum (over all the segments) the function value times the length of the segment, and then take the limit as the segment lengths approach zero.
  - Definition: For a curve C, a partition of C into n pieces is a list of points (x_0, y_0), ... , (x_n, y_n) on C, with the nth segment having length $\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. The norm of the partition P is the largest number among all of the segment lengths in P.
  - Definition: For $f(x, y)$ a continuous function and P a partition of the curve C, we define the Riemann sum of $f(x, y)$ on D corresponding to P to be $\mathrm{RS}_P(f) = \displaystyle\sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k$.
  - Definition: For a function $f(x, y)$, we define the line integral of f on the curve C, denoted $\displaystyle\int_C f(x, y)\, ds$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\mathrm{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.
  - Remark: It can be proven (with significant effort) that, if $f(x, y)$ is continuous and the curve C is smooth, then a value of L satisfying the hypotheses actually does exist.
  - Remark: The differential ds in the definition of the line integral is the "differential of arclength", which we discussed earlier in our study of vector-valued functions.

- In exactly the same way, we can use Riemann sums to give a formal definition of the line integral along a curve C in 3-space. (We simply add the appropriate z-terms to all the definitions.)

- Like with the other types of integrals, line integrals have a number of formal properties which can be deduced from the Riemann sum definition. Specifically, for D an arbitrary constant and $f(x, y)$ and $g(x, y)$ continuous functions, the following properties hold:
  - Integral of constant: $\int_C D\, ds = D \cdot \mathrm{Arclength}(C)$.
  - Constant multiple of a function: $\int_C D\, f(x, y)\, ds = D \cdot \int_C f(x, y)\, ds$.
  - Addition of functions: $\int_C f(x, y)\, ds + \int_C g(x, y)\, ds = \int_C [f(x, y) + g(x, y)]\, ds$.
  - Subtraction of functions: $\int_C f(x, y)\, ds - \int_C g(x, y)\, ds = \int_C [f(x, y) - g(x, y)]\, ds$.
  - Nonnegativity: if $f(x, y) \ge 0$, then $\int_C f(x, y)\, ds \ge 0$.
  - Union: If C_1 and C_2 are curves such that C_2 starts where C_1 ends, and C is the curve obtained by gluing the curves end-to-end, then $\int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds = \int_C f(x, y)\, ds$.
  - Remark: These same properties also all hold for line integrals of a function $f(x, y, z)$ in 3-space.

- The key observation is that we can reduce calculations of line integrals to "traditional" single integrals:
- Proposition (Line Integrals in the Plane): If the curve C can be parametrized as $x = x(t)$, $y = y(t)$ for $a \le t \le b$, then $\displaystyle\int_C f(x,y)\,ds = \int_a^b f(x(t),\,y(t))\,\frac{ds}{dt}\,dt$, where $\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$ is the derivative of arclength.
- Proposition (Line Integrals in 3-Space): If the curve C can be parametrized as $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, then $\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(x(t),\,y(t),\,z(t))\,\frac{ds}{dt}\,dt$, where $\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ is the derivative of arclength.
  - The proof of both of these results is simply to observe that the Riemann sum $\displaystyle\sum_{k=1}^{n} f(x_k,y_k)\,\Delta s_k$ for the line integral $\int_C f(x,y)\,ds$ is also a Riemann sum $\displaystyle\sum_{k=1}^{n} f(x_k,y_k)\,\frac{\Delta s_k}{\Delta t_k}\Delta t_k$ for the integral $\int_a^b f(x(t),\,y(t))\,\dfrac{ds}{dt}\,dt$.
  - Equivalently: we have made a substitution in the integral by changing from s-coordinates to t-coordinates, where the differential changes using the rule $ds = \dfrac{ds}{dt}dt$.
- Thus, to evaluate the line integral of f on the curve C (i.e., the line integral $\int_C f(x,y,z)\,ds$), follow these steps:
  1. Parametrize the curve C as a function of t, as $\mathbf{r}(t) = \langle x(t),\,y(t),\,z(t)\rangle$ for $a \le t \le b$.
  2. Write the function f in terms of t: $f(x,y,z) = f(x(t),y(t),z(t))$.
  3. Write the differential $ds = \dfrac{ds}{dt}\,dt = ||\mathbf{v}(t)||\,dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$ in terms of t.
  4. Evaluate the resulting single-variable integral $\int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$.
- Example: Integrate the function $f(x,y,z) = yz - 6x$ along the curve $\mathbf{r}(t) = \left\langle t^3, 6t, 3t^2\right\rangle$ from $t = 0$ to $t = 1$.
  - We have $f(x,y,z) = yz - 6x = (6t)(3t^2) - 6t^3 = 12t^3$, and we also have $ds = \sqrt{(3t^2)^2 + (6)^2 + (6t)^2} = \sqrt{9t^4 + 36t^2 + 36} = 3t^2 + 6$.
  - The integral is therefore $\int_0^1 (12t^3)(3t^2+6)dt = \int_0^1 (36t^5 + 72t^3)\,dt = \boxed{24}$.
- Example: Integrate the function $f(x,y) = x^2 + y$ along the top half of the unit circle $x^2 + y^2 = 1$, starting at $(1,0)$ and ending at $(-1,0)$.
  - The unit circle is parametrized by $\mathbf{r}(t) = \langle \cos t,\,\sin t\rangle$: the range we want is $0 \le t \le \pi$.
  - We have $f(x,y) = x^2 + y = \cos^2 t + \sin t$, and we also have $ds = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.
  - The integral is therefore $\int_0^\pi \left[\cos^2 t + \sin t\right]dt = \int_0^\pi \left[\dfrac{1+\cos 2t}{2} + \sin t\right]dt = \boxed{\dfrac{\pi}{2} + 2}$.
- To find the average value of a function on a curve, we simply integrate the function over the curve, and then divide by the curve's arclength.
- Example: Find the average value of the function $f(x,y,z) = x^2 + y^2 + z^2$ along the line segment from $(1,-1,0)$ to $(2,2,1)$.
  - The direction vector for the line is $\mathbf{v} = \langle 2,2,1\rangle - \langle 1,-1,0\rangle = \langle 1,3,1\rangle$. Thus, we can parametrize the line segment as $\langle x,y,z\rangle = \langle 1,-1,0\rangle + t\,\langle 1,3,1\rangle$ for $0 \le t \le 1$.
  - So the line segment is parametrized explicitly by $x = 1+t$, $y = -1+3t$, $z = t$ for $0 \le t \le 1$.
  - Now we set up the integral: the function is $f(x,y,z) = x^2 + y^2 + z^2 = (1+t)^2 + (-1+3t)^2 + (t)^2 = 11t^2 - 4t + 2$.
  - Since $x'(t) = 1$, $y'(t) = 3$, and $z'(t) = 1$, we also have $\dfrac{ds}{dt} = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$.

- The integral of f is therefore $\int_0^1 \left[11t^2 - 4t + 2\right] \sqrt{11} dt = \sqrt{11} \left[\frac{11}{3}t^3 - 2t^2 + 2t\right]\Big|_{t=0}^{1} = \frac{11\sqrt{11}}{3}$.
- To compute the average value, we divide by the arclength, which is $\int_0^1 1\, ds = \int_0^1 \sqrt{11} dt = \sqrt{11}$.
- Thus, the average value is $\boxed{\frac{11}{3}}$.

- We also have formulas for the mass and moments of a wire of variable density:
- <u>Center of Mass and Moment Formulas</u> (Thin Wire): Given a 1-dimensional wire of variable density $\delta(x, y, z)$ along a parametric curve C in 3-space:
  - The total mass M is given by $M = \int_C \delta(x, y, z)\, ds$.
  - The x-moment M_{yz} is given by $M_{yz} = \int_C x\, \delta(x, y, z)\, ds$.
  - The y-moment M_{xz} is given by $M_{xz} = \int_C y\, \delta(x, y, z)\, ds$.
  - The z-moment M_{xy} is given by $M_{xy} = \int_C z\, \delta(x, y, z)\, ds$.
  - The center of mass $(\bar{x}, \bar{y}, \bar{z})$ has coordinates $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$.
  - <u>Note</u>: For a wire in 2-space, the formulas are essentially the same (except without the z-coordinate), though the x-moment is denoted M_y and the y-moment is denoted M_x.

- <u>Example</u>: Find the total mass, and the center of mass, of a thin wire in the xy-plane having the shape of the unit circle with variable density $\delta(x, y) = 2 + x$.
  - We can parametrize the unit circle with $x = \cos t$, $y = \sin t$, so $\frac{ds}{dt} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.
  - The total mass M is $M = \int_C \delta(x, y)\, ds = \int_0^{2\pi} (2 + \cos t)\, dt = \boxed{2\pi}$.
  - The x-moment M_y is $M_y = \int_C x\, \delta(x, y)\, ds = \int_0^{2\pi} \cos t(2 + \cos t)\, dt = \left[2\sin t + \frac{1}{2}t + \frac{1}{4}\sin(2t)\right]\Big|_{t=0}^{2\pi} = \pi$.
  - The y-moment M_x is $M_x = \int_C y\, \delta(x, y)\, ds = \int_0^{2\pi} \sin t(2 + \cos t)\, dt = \left[-2\cos t - \frac{1}{4}\cos(2t)\right]\Big|_{t=0}^{2\pi} = 0$.
  - Therefore, the center of mass is $\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{1}{2}, 0\right)}$.

- We will also be interested in computing line integrals involving the differentials dx, dy, and dz rather than ds: namely, expressions of the form $\int_C f\, dx + g\, dy + h\, dz$.
- We evaluate such line integrals by making the appropriate substitutions: if C is parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, then the line integral $\int_C f\, dx + g\, dy + h\, dz$ is given by the single-variable integral $\int_a^b \left[f\,\frac{dx}{dt} + g\,\frac{dy}{dt} + h\,\frac{dz}{dt}\right] dt$.
- <u>Example</u>: Find $\int_C y\, dx + z\, dy + x^2\, dz$, where C is the curve $(x, y, z) = (t, t^2, t^3)$ ranging from $t = 0$ to $t = 1$.
  - We have $x = t$, $y = t^2$, and $z = t^3$, so that $dx = dt$, $dy = 2t\, dt$, and $dz = 3t^2\, dt$.
  - The integral is $\int_0^1 \left[t^2 \cdot dt + 3t^2 \cdot 2t\, dt + t^2 \cdot 3t^2\, dt\right] = \int_0^1 \left[t^2 + 6t^3 + 3t^4\right] dt = \boxed{\frac{73}{30}}$.

- <u>Example</u>: Find $\int_C x\, dy - y\, dx$, where C is the upper half of the ellipse $x^2/9 + y^2/16 = 1$, starting at $(3, 0)$ and ending at $(-3, 0)$.
  - This ellipse is parametrized by $\mathbf{r}(t) = \langle 3\cos t, 4\sin t\rangle$: the range we want is $0 \le t \le \pi$.
  - We have $x = 3\cos t$ and $y = 4\sin t$, so that $dx = -3\sin t\, dt$ and $dy = 4\cos t\, dt$.
  - The desired integral is $\int_0^\pi \left[3\cos t \cdot (4\cos t\, dt) - 4\sin t \cdot (-3\sin t\, dt)\right] = \int_0^\pi \left[12\cos^2 t + 12\sin^2 t\right] dt = \boxed{12\pi}$.

4.2 Surfaces and Surface Integrals

- We would now like to consider the problem of computing the integral of a function on a surface in 3-dimensional space. In a similar way to how we computed line integrals using (single) integrals, we will be able to compute surface integrals as double integrals.

- There are essentially two ways to describe a surface in 3-space: either as an implicit surface of the form $f(x, y, z) = c$, or as a parametric surface $\mathbf{r}(s, t) = \langle x(s, t),\, y(s, t),\, z(s, t)\rangle$ for two parameters s and t.

  - Note that the "explicit surface" $z = g(x, y)$ is simply a special case of the general implicit surface, since $g(x, y) - z = 0$ has the form $f(x, y, z) = c$ with $f(x, y, z) = g(x, y) - z$ and $c = 0$.
  - In cases where the functions x, y, and z are sufficiently simple or nice, it can be possible to eliminate the variables s and t from the system $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$, and obtain an equation for the surface as an implicit surface $f(x, y, z) = c$.
  - We will also remark that parametric descriptions of surfaces are often easier to work with than implicit descriptions. For example, graphing a parametric surface requires only plugging in values for (s, t) and plotting the resulting points (x, y, z), whereas graphing an implicit surface requires finding solutions to the implicit equation, which is typically much harder.

- We will describe how to find parametrizations of some common surfaces, give the definition of a surface integral, and then show how to compute surface integrals on both parametric and implicit surfaces.

4.2.1 Parametric Surfaces

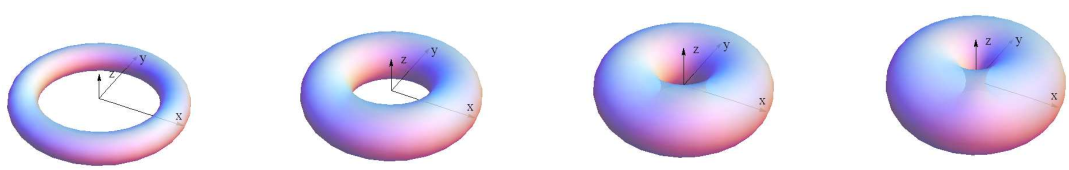
- If we graph a vector-valued function of two variables $\mathbf{r}(s, t) = \langle x(s, t),\, y(s, t),\, z(s, t)\rangle$ as s and t vary, we will obtain a surface in space (barring something strange happening).

- Example: The surface $\mathbf{r}(s, t) = \langle x_0, y_0, z_0\rangle + t\,\langle v_1, v_2, v_3\rangle + s\,\langle w_1, w_2, w_3\rangle$ is the plane passing through the point (x_0, y_0, z_0) that contains the two vectors $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$, provided that $\mathbf{v}$ and $\mathbf{w}$ are not parallel.

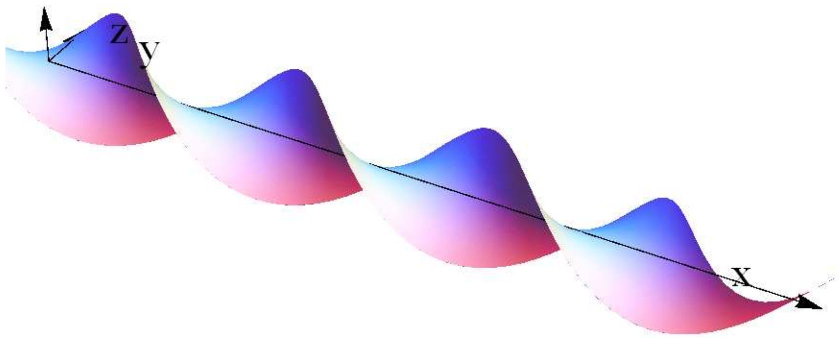
  - We could also describe the plane as an implicit surface of the form $ax+by+cz = d$, where $\langle a, b, c\rangle = \mathbf{v}\times\mathbf{w}$ is the normal vector to the plane and $d = ax_0 + by_0 + cz_0$.
  - There are many ways to describe a given plane as a parametric surface. For example, both of the parametrizations $\mathbf{r}(s, t) = \langle s,\, t,\, 1 - s - t\rangle$ and $\mathbf{r}(s, t) = \langle -3 + s - 2t,\, 2 + t + 2s,\, 2 + t - 3s\rangle$ describe the same plane $x + y + z = 1$.

- Example: For two positive "radius parameters" r and R with $r < R$, the surface defined parametrically by $\mathbf{r}(s, t) = \langle \cos(t)\cdot[R + r\sin(s)],\, \sin(t)\cdot[R + r\cos(s)],\, r\sin(s)\rangle$, for $0 \le t \le 2\pi$ and $0 \le s \le 2\pi$ is a donut-shaped surface called a torus.

  - It is the surface obtained by taking a vertical circle of radius r and moving its center along the circle $x^2 + y^2 = R^2$ in the xy-plane.
  - Four tori, with respective parameters (r, R) equal to $(1, 5)$, $(2, 5)$, $(3, 5)$, and $(4, 5)$, are plotted below:



- Example: The surface defined parametrically by $\mathbf{r}(s,t) = \langle \cos(s) + \cos(t),\ s+t,\ \sin(s) + \sin(t) \rangle$, for $0 \le t \le 4\pi$ and $0 \le s \le 4\pi$ is a helical ribbon:



- In general, it can be a somewhat involved problem to convert a geometric or verbal description of a surface into a parametrization: it is really more of an art form than a general procedure.
  - To parametrize parts of cylinders, cones, and spheres, it is almost always a very good idea to consider whether cylindrical or spherical coordinates can be of assistance.
  - Using translations and rescalings, we can also parametrize surfaces like ellipsoids.

- There are many different ways to parametrize the same surface, and which description is best will depend on what the parametrization will be used for.
  - For example, $x = s$, $y = t$, $z = \sqrt{s^2 + t^2}$ parametrizes the cone $z = \sqrt{x^2 + y^2}$, but so does the parametrization $x = s\cos t$, $y = s \sin t$, $z = s$.
  - If we want to describe the points lying over a rectangular region in the xy-plane, the first parametrization is more useful, but if we want to describe the points on the cone up to a specific height in the z-direction, the second parametrization is more useful.

- Example: Parametrize the portion of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = -2$ and $z = 2$.
  - In cylindrical coordinates, we know that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.
  - Since the given cylinder has equation $r = 2$ in cylindrical coordinates, we see that a parametrization of the full cylinder is $x = 2\cos t$, $y = 2\sin t$, $z = s$, where $0 \le t \le 2\pi$ but with no restrictions on s. (Here we think of t as θ and s as z.)
  - To obtain just the portion with $-2 \le z \le 2$ we just restrict the range for s.
  - Thus the parametrization of the desired portion of the cylinder is $x = 2\cos t$, $y = 2\sin t$, $z = s$, where $0 \le t \le 2\pi$ and $-2 \le s \le 2$.

- Example: Parametrize the portion of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = y - 2$ and $z = x + 4$.
  - Like in the previous example, we take the parametrization of the full cylinder as $x = 2\cos t$, $y = 2\sin t$, $z = s$, and then restrict the ranges for s and t appropriately. In this case, we want the portion of the surface where $y - 2 \le z \le x + 4$.
  - It is straightforward to check that the two planes do not intersect inside the cylinder (since $y - 2 \le 0$ inside the cylinder, while $x + 4 \ge 2$).
  - So in this case, we take $0 \le t \le 2\pi$ and $2\sin t \le s \le 2\cos t + 4$.

- Example: Parametrize the sphere $x^2 + y^2 + z^2 = 9$.
  - In spherical coordinates, we know that $x = \rho\,\cos(\theta)\sin(\varphi)$, $y = \rho\,\sin(\theta)\,\sin(\varphi)$, $z = \rho\,\cos(\varphi)$.

- The sphere has equation $\rho = 3$, so we can immediately see that $x = 3\cos(t)\sin(s)$, $y = 3\sin(t)\sin(s)$, $z = 3\cos(s)$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$, will parametrize the sphere. (Here, we are thinking of t as θ and s as φ.)

- Example: Parametrize the sphere $(x-2)^2 + (y+1)^2 + (z-6)^2 = 4$.
  - It is not so easy to describe this sphere using spherical coordinates directly. However, if we shift the coordinates to center the sphere at the origin, we can easily write down the parametrization.
  - By translating back, we can see that $x = 2 + 2\cos(t)\sin(s)$, $y = -1 + 2\sin(t)\sin(s)$, $z = 6 + 2\cos(s)$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$, will parametrize the sphere.

- Example: Parametrize the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$.
  - It is again not so easy to write down the parametrization using any of our coordinate systems directly. However, if we rescale the coordinates by setting $x' = x/2$, $y' = y/3$, and $z' = z/4$, then we see $(x')^2 + (y')^2 + (z')^2 = 1$, and we can use spherical coordinates to parametrize the coordinates x', y', z'.
  - By rescaling back, we can see that $x = 2\cos(t)\sin(s)$, $y = 3\sin(t)\sin(s)$, $z = 4\cos(s)$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$, will parametrize this ellipsoid.

- Example: Parametrize the portion of the cone $z = 3\sqrt{x^2+y^2}$ that lies below the plane $z = 1 + x + y$.
  - In cylindrical, the equations are $z = 3r$ and $z = 2 + r\cos\theta + r\sin\theta$. They are equal when $3r = 2 + r\cos\theta + r\sin\theta$, or $r = \dfrac{2}{3 - \cos\theta - \sin\theta}$. (Note that $\sin\theta + \cos\theta \le \sqrt{2}$, so the denominator is never zero.)
  - The full surface is parametrized by $x = s\cos(t)$, $y = s\sin(t)$, $z = 3s$.
  - The portion under the plane corresponds to $0 \le s \le \dfrac{2}{3 - \cos t - \sin t}$, with $0 \le t \le 2\pi$.

- If we have a parametrization of a surface, we can use the parametrization to find the tangent plane to the surface at a given point.
  - The key observation is that if the surface S is parametrized by the vector-valued function $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$, then the two partial derivatives $\mathbf{r}_s = \dfrac{\partial \mathbf{r}}{\partial s}$ and $\mathbf{r}_t = \dfrac{\partial \mathbf{r}}{\partial t}$ are both tangent to the surface.
  - Therefore, the cross product $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$ will be perpendicular to the tangent plane, and is thus a normal vector for the tangent plane.

- Example: Find an equation for the tangent plane to the surface $\mathbf{r}(s,t) = \left\langle s\cos(t), s\sin(t), s^2\right\rangle$ when $s = 1$ and $t = \pi/2$.
  - We compute $\mathbf{r}_s(s,t) = \langle \cos t, \sin t, 2s\rangle$ and $\mathbf{r}_t(s,t) = \langle -s\sin t, s\cos t, 0\rangle$.
  - Thus, we see $\mathbf{r}_s(1,\pi/2) = \langle 0,1,2\rangle$, and $\mathbf{r}_t(1,\pi/2) = \langle -1,0,0\rangle$, and so the normal vector to the tangent plane is $\mathbf{n} = \langle 0,1,2\rangle \times \langle -1,0,0\rangle = \langle 0,-2,1\rangle$.
  - The tangent plane passes through the point on the surface where $s = 1$ and $t = \pi/2$, which is $\mathbf{r}(1,\pi/2) = \langle 0,1,1\rangle$.
  - Thus, an equation for the tangent plane is given by $\boxed{0(x-0) - 2(y-1) + 1(z-1) = 0}$ or equivalently $\boxed{-2y + z = -1}$.

- Example: Find an equation for the plane tangent to the surface $\mathbf{r}(s,t) = \left\langle s^2, 2st, t^3\right\rangle$ at the point $(4,4,-1)$.
  - First, we need to find the values of s and t at the point $(4,4,-1)$. If $\langle 4,4,-1\rangle = \left\langle s^2, 2st, t^3\right\rangle$ then we see $t^3 = -1$ so $t = -1$, and then $2st = 4$ gives $s = -2$.

- Now, we have $\mathbf{r}_s(s,t) = \langle 2s, 2t, 0 \rangle$ and $\mathbf{r}_t(s,t) = \left\langle 0, 2s, 3t^2 \right\rangle$, so $\mathbf{r}_s(-2,-1) = \langle -4, -2, 0 \rangle$ and $\mathbf{r}_t(-2,-1) = \langle 0, -4, 3 \rangle$.
- Thus, the normal vector to the tangent plane is $\mathbf{n} = \langle -4, -2, 0 \rangle \times \langle 0, -4, 3 \rangle = \langle -6, 12, 16 \rangle$.
- Thus, an equation for the tangent plane is given by $\boxed{-6(x-4) + 12(y-4) + 16(z+1) = 0}$ or equivalently $\boxed{-6x + 12y + 16z = 8}$.

4.2.2 Surface Integrals

- The motivating problem for our discussion of surface integrals is as follows: given a parametric surface $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t) \rangle$ and a function $f(x,y,z)$, we would like to integrate the function on that surface. Like with line integrals, we have two natural applications: computing the average value of a function on the surface, and analyzing the physical properties of a thin surface with variable density.
  - As with all the other types of integrals, the idea is to approximate the surface with small "patches", sum (over all the patches) the function value times the area of the patch, and then take the limit as the patch sizes approach zero.
  - Definition: For a parametric surface S, a partition of S into n pieces is a list of disjoint subregions inside S, where the kth subregion corresponds to $s_k \le s \le s'_k$, $t_k \le t \le t'_k$, and has area $\Delta\sigma_k$. The norm of the partition P is the largest number among the areas of the rectangles in P.
  - Definition: For $f(x,y,z)$ a continuous function and P a partition of the surface S, we define the Riemann sum of $f(x,y,z)$ on R corresponding to P to be $\text{RS}_P(f) = \displaystyle\sum_{k=1}^{n} f(\mathbf{r}(s_k, t_k))\, \Delta\sigma_k$.
  - Definition: For a function $f(x,y,z)$, we define the surface integral of f on S, denoted $\displaystyle\iint_S f(x,y,z)\, d\sigma$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\text{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.
  - Remark: It can be proven (with significant effort) that, if $f(x,y,z)$ is continuous, then a value of L satisfying the hypotheses actually does exist.
- As with all of the other types of integrals, surface integrals possess some formal properties:
  - Integral of constant: $\iint_S C\, d\sigma = C \cdot \text{Area}(S)$.
  - Constant multiple of a function: $\iint_S C\, f(x,y)\, d\sigma = C \cdot \iint_S f(x,y)\, d\sigma$.
  - Addition of functions: $\iint_S f(x,y)\, d\sigma + \iint_S g(x,y)\, d\sigma = \iint_S [f(x,y) + g(x,y)]\, d\sigma$.
  - Subtraction of functions: $\iint_S f(x,y)\, d\sigma - \iint_S g(x,y)\, d\sigma = \iint_S [f(x,y) - g(x,y)]\, d\sigma$.
  - Nonnegativity: if $f(x,y) \ge 0$, then $\iint_S f(x,y)\, d\sigma \ge 0$.
  - Union: If S_1 and S_2 don't overlap and have union S, then $\iint_{S_1} f(x,y)\, d\sigma + \iint_{S_2} f(x,y)\, d\sigma = \iint_S f(x,y)\, d\sigma$.
- We were able to reduce line integral calculations to standard one-variable integrals. We can similarly reduce calculations of surface integrals to double integrals:
- Proposition (Parametric Surface Integrals): If $f(x,y,z)$ is continuous on the surface S which is parametrized as $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t) \rangle$, where S is described by a region R in st-coordinates, then the surface integral of f on S is

$$\iint_S f(x,y,z)\, d\sigma \quad = \quad \iint_R f(x(s,t),\, y(s,t),\, z(s,t)) \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| dt\, ds.$$

  - The key step is to recognize the Riemann sum for the surface integral as the Riemann sum for a particular double integral.

- Ultimately, the differential of surface area $d\sigma = \left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| dt\, ds$ arises from computing the area of a small patch in st-coordinates: when s changes slightly, the change in $\mathbf{r}$ is given by $\frac{\partial \mathbf{r}}{\partial s}$, and when t changes slightly, the change in $\mathbf{r}$ is given by $\frac{\partial \mathbf{r}}{\partial t}$.
- These two vectors form a small parallelogram that closely approximates the surface S, so the differential of surface area $d\sigma$ is roughly equal to the area of this parallelogram, which is $\left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right|$, times the differential $dt\, ds$.

- We can also calculate surface integrals over implicit surfaces of the form $g(x, y, z) = c$:
- Proposition (Implicit Surface Integrals): If $f(x, y, z)$ is continuous on the implicit surface S defined by $g(x, y, z) = c$, R is the projection of S into the xy-plane, and $\partial g/\partial z \neq 0$ on R, then the surface integral of f on S is

$$\iint_S f(x, y, z)\, d\sigma \quad = \quad \iint_R f(x, y, z) \frac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx$$

where ∇g is the gradient of g and $\mathbf{k} = \langle 0, 0, 1\rangle$. (Thus, $\nabla g \cdot \mathbf{k} = \partial g/\partial z$.)

  - The statement that $\partial g/\partial z \neq 0$ on R is equivalent to saying that the tangent plane to $g(x, y, z) = c$ is never vertical above R. In particular this implies that the surface never "doubles back" on itself over the region R.
  - Thus for example, we could not use the method directly to compute a surface integral on the entire unit sphere, because it has a vertical tangent plane above its projection $x^2 + y^2 \leq 1$ in the xy-plane.
  - This formula can be derived from the parametric surface integral formula: after some simplification, it is what one obtains by using the parametrization $\mathbf{r}(s, t) = \langle s, t, z(s, t)\rangle$, where $z(s, t)$ is defined implicitly via the relation $f(s, t, z(s, t)) = c$.

- Using these two results, we can reduce calculations of surface integrals to "traditional" double integrals: given a description of the surface S, we can convert it to a double integral using one of two methods:
  - For a parametric surface given in the form $\mathbf{r}(s, t) = \langle x(s, t),\ y(s, t),\ z(s, t)\rangle$:
    - Step 1: Find the bounds on s and t that parametrize the desired portion of the surface.
    - Step 2: Express the function $f(x, y, z)$ to be integrated in terms of (s, t).
    - Step 3: Find the differential of surface area $d\sigma = \left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| ds\, dt$.
    - Step 4: Write down the integral $\boxed{\iint_S f(x(s,t), y(s,t), z(s,t)) \left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| ds\, dt}$ and evaluate.
  - For an implicit surface given in the form $g(x, y, z) = c$:
    - Step 1: Sketch the surface, determine the shape of its projection R into the xy-plane, and make sure that the surface does not cover any part of the projection more than once.
    - Step 2: Evaluate the integral $\boxed{\iint_R f(x, y, z) \frac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx}$, where ∇g is the gradient of g and $\mathbf{k} = \langle 0, 0, 1\rangle$.
    - Note that the only variables allowed in the integral are x and y, so if the integrand has any z terms we must use the implicit equation $g(x, y, z) = c$ to get rid of them.
  - Note that, by swapping z with x or with y, the implicit surface procedure can also be used with a projection into the xz-plane or the yz-plane.
  - Also note that for a surface of the form $z = f(x, y)$, we could use either method.

- Example: Integrate the function $g(x, y, z) = z$ over the surface with parametrization $\mathbf{r}(s, t) = \langle \sin(t),\ \cos(t),\ s + t\rangle$ for $0 \leq t \leq 2\pi$ and $0 \leq s \leq \pi$.

- We have an explicit parametrization of the surface, so we use the parametric formula.
- On the surface, we have $z = s + t$ so $g(x, y, z) = z = s + t$.
- We have $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 0, 0, 1 \rangle$ and $\dfrac{\partial \mathbf{r}}{\partial t} = \langle \cos(t), -\sin(t), 1 \rangle$, so $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos(t) & -\sin(t) & 1 \end{vmatrix} = \langle \sin(t), \cos(t), 0 \rangle$.

  Then $\left|\left| \dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} \right|\right| = 1$.
- The integral is therefore given by

$$\int_0^{2\pi} \int_0^{\pi} (s+t)\, ds\, dt \quad = \quad \int_0^{2\pi} \left[\frac{s^2}{2} + st \right] \Big|_{s=0}^{\pi} = dt \int_0^{2\pi} \left[\frac{\pi^2}{2} + \pi t \right] dt = \left[\frac{\pi^2}{2} t + \frac{\pi}{2} t^2 \right] \Big|_{t=0}^{2\pi} = \boxed{3\pi^3}.$$

- Example: Integrate the function $f(x, y, z) = 8xy$ over the portion of the plane $2x + y + 2z = 1$ with $0 \le x \le 1$, $0 \le y \le 1$.
  - We use the implicit surface formula, with $g(x, y, z) = 2x + y + 2z - 1$.
  - We have $\nabla g = \langle 2, 1, 2 \rangle$ so $||\nabla g|| = \sqrt{2^2 + 1^2 + 2^2} = 3$ and $|\nabla g \cdot \mathbf{k}| = 2$.
  - The desired integral is therefore $\int_0^1 \int_0^1 8xy \cdot (3/2)\, dy\, dx = \int_0^1 6x\, dx = \boxed{3}$.
- Example: Integrate the function $f(x, y, z) = xz$ over the portion of the plane $4x + 2y + z = 1$ with $0 \le x \le 1$, $0 \le y \le 1$.
  - We use the implicit surface formula, with $g(x, y, z) = 4x + 2y + z - 1$.
  - We have $\nabla g = \langle 4, 2, 1 \rangle$ so $||\nabla g|| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$ and $|\nabla g \cdot \mathbf{k}| = 1$.
  - Since the function involves z, we must use the implicit relation to eliminate it. In this case, $z = 1 - 4x - 2y$, so $f(x, y, z) = xz = x - 4x^2 - 2xy$.
  - The desired integral is therefore $\int_0^1 \int_0^1 (x - 4x^2 - 2xy) \cdot \sqrt{21}\, dy\, dx = \int_0^1 (-4x^2) \sqrt{21}\, dx = \boxed{-\frac{4}{3}\sqrt{21}}$.
- To compute surface area, we can simply integrate the function 1 on the surface, in exactly the same way that integrating 1 on a plane region gives its area or integrating 1 on a solid region gives its volume.
- Example: Find the area of the portion of the surface $z = 2 - x^2 - y^2$ that lies above the xy-plane.
  - We can rewrite the equation of the surface "implicitly" as $x^2 + y^2 + z - 2 = 0$, so we use the implicit surface formula.
  - The projection of the surface into the xy-plane is the region R on which $2 - x^2 - y^2 \ge 0$, which is the same as $x^2 + y^2 \le 2$, and this describes the disc of radius $\sqrt{2}$ centered at the origin. Since this surface is explicit we do not need to worry about having a vertical tangent plane.
  - We have $\nabla g = \langle 2x, 2y, 1 \rangle$ so $||\nabla g|| = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla g \cdot \mathbf{k}| = 1$. The desired integral is therefore $\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dy\, dx$, since to calculate surface area we simply integrate the function 1.
  - To evaluate this integral, we change to polar coordinates, since both the region and the function to be integrated will become simpler: the region is $0 \le r \le \sqrt{2}$, $0 \le \theta \le 2\pi$, and the function is $\sqrt{4r^2 + 1}$.
  - Since the area differential in polar is $r\, dr\, d\theta$, we obtain the polar integral $\int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta$.
  - To evaluate this new integral, we make (another) substitution $u = 4r^2 + 1$, with $du = 8r\, dr$:

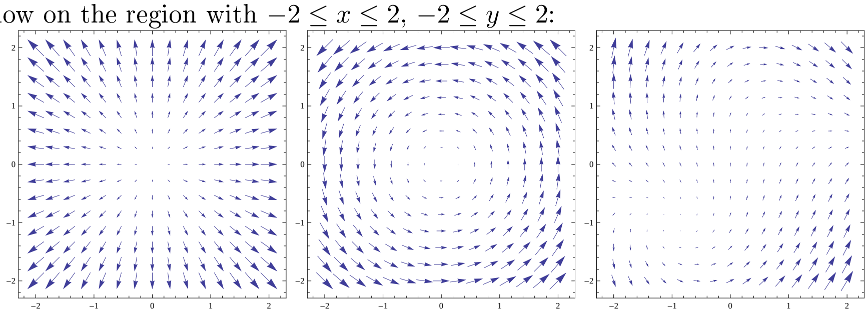
$$\int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta \quad = \quad \int_0^{2\pi} \int_1^{9} \frac{1}{8} \sqrt{u}\, du\, d\theta = \int_0^{2\pi} \frac{1}{8} \left(\frac{2}{3} u^{3/2} \right) \Big|_{u=1}^{9} d\theta = \int_0^{2\pi} \frac{26}{12}\, d\theta = \boxed{\frac{13\pi}{3}}.$$

  - Remark: Alternatively, we could have parametrized this surface using cylindrical coordinates, as $x = s\cos(t)$, $y = s\sin(t)$, $z = 2 - s^2$ for $0 \le s \le \sqrt{2}$, $0 \le t \le 2\pi$. This would have led us directly to the integral that showed up at the end (with s and t in place of r and θ).

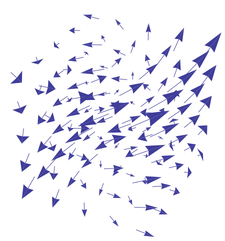
- To find the average value of a function on a surface, we integrate the function on the surface and then divide by the surface area.
- Example: Find the average value of $f(x, y, z) = z$ on the surface S given by the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 4$.
  - By using cylindrical coordinates we see that we can parametrize this portion of the cone as $x = s\cos(t)$, $y = s\sin(t)$, $z = s$, for $0 \le s \le 2$ and $0 \le t \le 2\pi$.
  - We then have $\mathbf{r}(s,t) = \langle s\cos(t),\, s\sin(t),\, s\rangle$, so $\dfrac{d\mathbf{r}}{ds} = \langle \cos(t),\, \sin(t),\, 1\rangle$ and $\dfrac{d\mathbf{r}}{dt} = \langle -s\sin(t),\, s\cos(t), 0\rangle$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & \sin(t) & 1 \\ -s\,\sin(t) & s\,\cos(t) & 0 \end{vmatrix} = \langle -s\cos(t),\, s\sin(t),\, s\rangle$, so the magnitude is given by $\left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right| = \sqrt{s^2\cos^2(t) + s^2\sin^2(t) + s^2} = s\sqrt{2}$.
  - We also have $f(x, y, z) = z = s$. So $\iint_S z\, d\sigma = \int_0^{2\pi}\int_0^2 s \cdot s\sqrt{2}\, ds\, dt = \int_0^{2\pi} \frac{8}{3}\sqrt{2}\, dt = \dfrac{16\pi\sqrt{2}}{3}$.
  - Also, the surface area is $\iint_S 1\, d\sigma = \int_0^{2\pi}\int_0^2 s\sqrt{2}\, ds\, dt = \int_0^{2\pi} 2\sqrt{2}\, dt = 4\pi\sqrt{2}$.
  - Thus, the average value is $\dfrac{1}{\text{Area}} \iint_S z\, d\sigma = \dfrac{16\pi\sqrt{2}/3}{4\pi\sqrt{2}} = \boxed{\dfrac{4}{3}}$.
- Like with double, triple, and line integrals, we have mass and moment formulas for surface integrals:
- Center of Mass and Moment Formulas (Thin Surface): Given a surface S of variable density $\delta(x, y, z)$ in 3-space:
  - The total mass M is given by $M = \iint_S \delta(x, y, z)\, d\sigma$.
  - The x-moment M_{yz} is given by $M_{yz} = \iint_S x\,\delta(x, y, z)\, d\sigma$.
  - The y-moment M_{xz} is given by $M_{xz} = \iint_S y\,\delta(x, y, z)\, d\sigma$.
  - The z-moment M_{xy} is given by $M_{xy} = \iint_S z\,\delta(x, y, z)\, d\sigma$.
  - The center of mass $(\bar{x}, \bar{y}, \bar{z})$ has coordinates $\left(\dfrac{M_{yz}}{M}, \dfrac{M_{xz}}{M}, \dfrac{M_{xy}}{M}\right)$.
- Example: A hill is shaped like the portion of the paraboloid $z = 4 - x^2 - y^2$ with $z \ge 0$, with all coordinates measured in meters. Snow accumulates on the hill such that the density is $\sqrt{17 - 4z}$ grams per square meter at height z. Find the total amount of snow on the hill.
  - We are given the density of snow and want to compute the total mass, which (per the above) is given by the integral $\iint_S \sqrt{17 - 4z}\, d\sigma$ where S is the surface representing the hill.
  - By using cylindrical coordinates, we can parametrize the hill as $\mathbf{r}(r, \theta) = \left\langle r\cos(\theta),\, r\sin(\theta),\, 4 - r^2\right\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial r} = \langle \cos(\theta),\, \sin(\theta),\, -2r\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin(\theta),\, r\cos(\theta), 0\rangle$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\,\sin(\theta) & r\,\cos(\theta) & 0 \end{vmatrix} = \left\langle 2r^2\cos(\theta),\, 2r^2\sin(\theta),\, r\right\rangle$, so $\left|\left|\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta}\right|\right| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}$.
  - We also have $f(x, y, z) = \sqrt{17 - 4(4 - r^2)} = \sqrt{4r^2 + 1}$.
  - Hence the integral becomes $\int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1} \cdot r\sqrt{4r^2 + 1}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (r + 4r^3)\, dr\, d\theta = \int_0^{2\pi} 18\, d\theta = 36\pi$.
  - Thus, there are $\boxed{36\pi \text{ g}}$ of snow on the hill.

4.3 Vector Fields, Work, Circulation, Flux

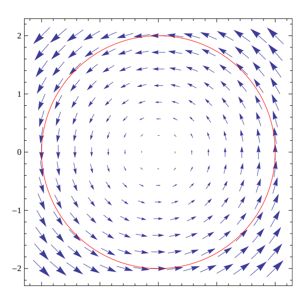
- Definition: A vector field in the plane is a function $\mathbf{F}(x, y) = \langle P(x, y),\, Q(x, y)\rangle$ that associates a vector to each point in the plane. A vector field in 3-space is a function $\mathbf{F}(x, y, z) = \langle P(x, y, z),\, Q(x, y, z),\, R(x, y, z)\rangle$ that associates a vector to each point in 3-space.
  - One vector field we have already encountered is the vector field associated to the gradient of a function $f(x, y)$ or $f(x, y, z)$: for example, if $f(x, y) = x^2 + xy$, then $\nabla f(x, y) = \langle 2x + y,\, x\rangle$.
- To represent a vector field visually, we choose some (nice) collection of points (generally in a grid) and draw the vectors corresponding to those points as arrows pointing in the appropriate direction and with the appropriate length.
  - Example: The three vector fields $\mathbf{F}(x, y) = \langle x, y\rangle$, $\mathbf{G}(x, y) = \langle -y, x\rangle$, and $\mathbf{H}(x, y) = \left\langle x + y^2,\, 2 - 2xy\right\rangle$ are plotted below on the region with $-2 \le x \le 2$, $-2 \le y \le 2$:



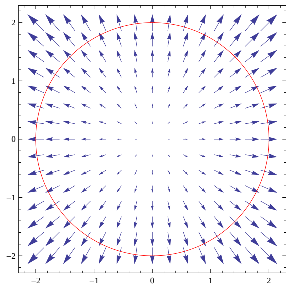
  - We can also produce these plots for 3-dimensional vector fields, but the diagrams tend to be quite cluttered; here is such a diagram for $\mathbf{F}(x, y, z) = \langle x, z - y, x + y\rangle$:



- We can think of a vector field as describing the flow of an incompressible fluid through space: the vector $\mathbf{F}(x, y)$ at any point (x, y) gives the direction and velocity of the fluid's flow there.
- In this context, if we have a particle that travels along some given path $\mathbf{r}(t)$ through the fluid, we might like to know how much work the fluid does on the particle, or (essentially equivalently) how much the fluid is pushing the particle along its path. This is the central idea behind work integrals and circulation integrals.
  - Intuitively, we see that the more the vector field $\mathbf{F}$ aligns with the tangent vector $\mathbf{T}$ to the particle's path, the more work it does.
  - In the picture, a particle moving counterclockwise around the circle will be pushed along its path by the vector field:



- Alternatively, if we have a particle traveling along a path, we could also ask: how much is the fluid pushing the particle off of the path? This is the central idea behind a flux integral.
  - Another way of thinking about this is to imagine the path as being a thin membrane, and asking how much fluid is passing across the membrane.
  - Here, we see that more fluid is flowing across the membrane if the vector field $\mathbf{F}$ aligns with the normal vector $\mathbf{N}$ to the particle's path:



- We can also formulate these ideas in 3-dimensional space: the ideas of circulation and work remain the same, but the notion of flux requires a surface for the fluid to flow across.

4.3.1 Circulation and Work Integrals

- To compute the circulation of a vector field along a curve, we want to integrate the quantity measuring how much the vector field is aligning with the path of motion along the curve.
- Definition: The (counterclockwise) circulation (or flow) of the vector field $\mathbf{F}$ along the curve C is defined to be $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{T}$ is the unit tangent to the curve.
  - What this says is: the circulation is given by integrating the dot product function $f(t) = \mathbf{F}(x(t),\, y(t)) \cdot \mathbf{T}(t)$ along the curve C. In order to evaluate the integral as written, we would need to parametrize C, find the unit tangent vector $\mathbf{T}(t)$ to the curve, and then integrate the dot product $\mathbf{F}(x(t),\, y(t)) \cdot \mathbf{T}(t)$ along the curve.
  - We would like to see if there is a simpler way, so let us suppose that $\mathbf{F}(x,y) = \langle P, Q\rangle$, where P and Q are functions of x and y, and say C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ from $t=a$ to $t=b$.
  - Then $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{||\mathbf{v}(t)||} = \dfrac{\left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}\right\rangle}{||\mathbf{v}(t)||}$, so $\mathbf{F}\cdot\mathbf{T} = \dfrac{\langle P, Q\rangle \cdot \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}\right\rangle}{||\mathbf{v}(t)||} = \dfrac{P\dfrac{dx}{dt} + Q\dfrac{dy}{dt}}{||\mathbf{v}(t)||}$.
  - We can then write $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \dfrac{P\dfrac{dx}{dt} + Q\dfrac{dy}{dt}}{||\mathbf{v}(t)||} \cdot ||\mathbf{v}(t)||\,dt = \int_a^b \left[P\dfrac{dx}{dt} + Q\dfrac{dy}{dt}\right] dt$.
  - Thus, the circulation integral can be written more explicitly as $\int_a^b \left[P\dfrac{dx}{dt} + Q\dfrac{dy}{dt}\right] dt$, where P, Q have been rewritten as functions of t. Note that this expression is also equal to $\int_C P\,dx + Q\,dy$.
  - We can also pose essentially the same definition for a curve in 3-space, and we obtain an analogous formula: if $\mathbf{F} = \langle P, Q, R\rangle$, then the circulation can be computed as $\int_a^b \left[P\dfrac{dx}{dt} + Q\dfrac{dy}{dt} + R\dfrac{dz}{dt}\right] dt$.
  - Terminology Note: Some authors reserve the term "circulation" for closed curves, and use "flow" to refer to the general case. This terminology can be somewhat confusing given that there is also a "flux" integral, and the words "flux" and "flow" (in non-technical settings) are synonyms.
- Example: Find the circulation of the vector field $\mathbf{G}(x,y) = \langle -y, x\rangle$ around a path that winds once counterclockwise around the unit circle.
  - We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$.
  - Thus, $P = -y = -\sin t$ and $Q = x = \cos t$, and also $\dfrac{dx}{dt} = -\sin t$ and $\dfrac{dy}{dt} = \cos t$.

- So, the circulation is $\int_a^b \left(P\,\frac{dx}{dt} + Q\,\frac{dy}{dt}\right)\,dt = \int_0^{2\pi}\left((-\sin t)(-\sin t) + (\cos t)(\cos t)\right)\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$.

- Example: Find the circulation of the vector field $\mathbf{F}(x, y, z) = \langle 2xy, xz, y\rangle$ along the line segment from $(0, 1, 0)$ to $(2, 2, 2)$.
  - We can parametrize the path as $x = 2t$, $y = 1 + t$, $z = 2t$ for $0 \le t \le 1$.
  - Thus, $P = 2xy = 4t + 4t^2$, $Q = xy = 4t^2$, and $R = 1 + t$.
  - So, the circulation is $\int_a^b \left(P\,\frac{dx}{dt} + Q\,\frac{dy}{dt} + R\,\frac{dz}{dt}\right)\,dt = \int_0^1 [(4t + 4t^2)\cdot 2 + 4t^2 \cdot 1 + (1 + t)\cdot 2]\,dt = \int_0^1 (2 + 10t + 12t^2)\,dt = \boxed{11}$.

- We can also pose a similar definition for the work done by a vector field on a particle:

- Definition: The work performed on a particle by a vector field $\mathbf{F}$ as the particle travels along a curve C is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$.
  - Note that the work integral has the same form as the circulation integral.
  - Notation: The "vector differential" $d\mathbf{r}$ is defined as $d\mathbf{r} = \langle dx, dy\rangle$ in the plane and as $d\mathbf{r} = \langle dx,\, dy,\, dz\rangle$ in 3-space.
  - Then $\mathbf{F}\cdot d\mathbf{r} = P\,dx + Q\,dy$, so the work integral is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \int_a^b \left[P\,\frac{dx}{dt} + Q\,\frac{dy}{dt}\right]\,dt$ in the plane, or as $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left[P\,\frac{dx}{dt} + Q\,\frac{dy}{dt} + R\,\frac{dz}{dt}\right]\,dt$ in 3-space.

- Example: Find the work done by the vector field $\mathbf{F}(x, y, z) = \langle 2x + z,\, yz,\, xy\rangle$ on a particle traveling along the path $\mathbf{r}(t) = \langle t,\, t^2,\, 2t\rangle$ from $t = 0$ to $t = 1$.
  - We have $P = 2x + z = 3t$, $Q = yz = 2t^3$, and $R = xy = t^3$. Also, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$, and $\frac{dz}{dt} = 2$.
  - Therefore, the work is $\int_a^b \left(P\,\frac{dx}{dt} + Q\,\frac{dy}{dt} + R\,\frac{dz}{dt}\right)\,dt = \int_0^1 \left[(3t)(1) + (2t^3)(2t) + (t^3)(2)\right]\,dt = \int_0^1 (3t + 4t^4 + 2t^3) = \boxed{\frac{14}{5}}$.

4.3.2 Flux Across a Curve

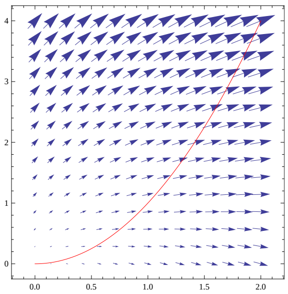
- To compute the flux of a vector field across a curve, we want to integrate the quantity measuring how much the vector field is moving in the direction perpendicular to the curve.

- Definition: The flux of the vector field $\mathbf{F}$ across the curve C is $\int_C \mathbf{F}\cdot\mathbf{N}\,ds$, where $\mathbf{N}$ is the unit normal to the curve.
  - As with the circulation integral, we would like an easier way to evaluate the flux integral.
  - If $\mathbf{F}(x, y) = \langle P, Q\rangle$ and C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ from $t = a$ to $t = b$, after some algebra we can calculate that $\mathbf{N}(t) = \frac{1}{||\mathbf{v}(t)||}\left\langle \frac{dy}{dt}, -\frac{dx}{dt}\right\rangle$. (At the very least, it is easy to observe that this is a unit vector that is orthogonal to $\mathbf{T}$.)
  - Then $\mathbf{F}\cdot\mathbf{N} = \dfrac{\langle P, Q\rangle\cdot\left\langle \frac{dy}{dt}, -\frac{dx}{dt}\right\rangle}{||\mathbf{v}(t)||} = \dfrac{P\,\frac{dy}{dt} - Q\,\frac{dx}{dt}}{||\mathbf{v}(t)||}$.
  - Plugging this in gives $\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_a^b \dfrac{P\,\frac{dy}{dt} - Q\,\frac{dx}{dt}}{||\mathbf{v}(t)||}\cdot ||\mathbf{v}(t)||\,dt = \int_a^b \left[P\,\frac{dy}{dt} - Q\,\frac{dx}{dt}\right]\,dt$.
  - Thus, the flux integral can be written more explicitly as $\int_C P\,dy - Q\,dx = \int_a^b \left[P\,\frac{dy}{dt} - Q\,\frac{dx}{dt}\right]\,dt$.

- Note: The flux integral as defined here only makes sense for curves in the plane. In 3-dimensional space, the corresponding notion requires a surface integral, since a "membrane" will be a surface, rather than a curve.

- Example: Find the flux of the vector field $\mathbf{G}(x, y) = \langle x, y\rangle$ across a path that winds once counterclockwise around the unit circle.
  - We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$.
  - Thus, $P = x = \cos t$ and $Q = y = \sin t$, and also $\dfrac{dx}{dt} = -\sin t$ and $\dfrac{dy}{dt} = \cos t$.
  - So Flux $= \int_a^b \left(P\,\dfrac{dy}{dt} - Q\,\dfrac{dx}{dt}\right) dt = \int_0^{2\pi} ((\cos t)(\cos t) - (\sin t)(-\sin t))\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$.

- Example: For the vector field $\mathbf{F}(x, y) = \langle 2x + y, 2y - x\rangle$, find the flux across, and circulation along, the portion of the curve $\mathbf{r}(t) = \langle t, t^2\rangle$ between $(0, 0)$ and $(2, 4)$.
  - Here is a plot of the vector field, along with the curve:



  - From the picture, we would expect the circulation and flux to be roughly equal, since the vector field makes roughly a 45-degree angle with the path near the end.
  - The parametrization given says $x = t$ and $y = t^2$, so that $P = 2x + y = 2t + t^2$ and $Q = 2y - x = 2t^2 - t$. Also, the start is $t = 0$ and the end is $t = 2$.
  - Then the circulation is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \left(P\,\dfrac{dx}{dt} + Q\,\dfrac{dy}{dt}\right) dt = \int_0^2 \left((2t + t^2)\cdot 1 + (2t^2 - t)\cdot 2t\right) dt = \int_0^2 \left(4t^3 - t^2 + 2t\right) dt = (t^4 - \dfrac{1}{3}t^3 + t^2)\Big|_{t=0}^{2} = \boxed{\dfrac{52}{3}}$.
  - The flux is $\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_a^b \left(P\,\dfrac{dy}{dt} - Q\,\dfrac{dx}{dt}\right) dt = \displaystyle\int_0^2 \left((2t + t^2)\cdot 2t - (2t^2 - t)\cdot 1\right) dt = \int_0^2 \left(2t^3 + 2t^2 + 2t\right) dt = (\dfrac{1}{2}t^4 + \dfrac{2}{3}t^3 + t^2)\Big|_{t=0}^{2} = \boxed{\dfrac{52}{3}}$.
  - Indeed, we see that the flux and circulation are roughly (and exactly) equal.

4.3.3 Flux Across a Surface

- In 3-space, the notion of circulation along a curve remains essentially the same as in the plane. However, in order to make sense of flux in 3-space, we must instead talk about flux through a surface rather than through a curve. This requires us to use a surface integral to measure how much the vector field is flowing across the surface:

- Definition: The (normal) flux of the vector field $\mathbf{F}$ across the surface S is $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$, where $\mathbf{n}$ is the outward unit normal to the surface.
  - Remark: The integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ computes the flux through the surface in the direction of the "outward normal". It is also possible to ask about flux in the direction of a particular unit vector $\mathbf{u}$; the integral in that case is $\iint_S \mathbf{F}\cdot\mathbf{u}\,d\sigma$, instead. In general, when it is not specified what type of flux integral is meant, the "flux in the direction of the outward normal" is intended.

- Recall that the normal vector to a surface is orthogonal to the tangent plane (it is in fact the normal vector to the tangent plane as we defined it earlier). When speaking of a unit normal to a surface we will use a lowercase $\mathbf{n}$, to keep the notation different from the unit normal $\mathbf{N}$ to a curve (which is an uppercase $\mathbf{N}$).
- If S is an implicit surface $g(x,y,z)=c$, then a normal vector is given by the gradient ∇g, so we get a unit normal vector $\mathbf{n} = \nabla g / \|\nabla g\|$.
- If S is parametrized by $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$, then a normal vector is given by the cross product $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$, so we get a unit normal vector $\mathbf{n} = \left(\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right) \Big/ \left\|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right\|$.
- Important Warning: If we scale the implicit equation by -1, or write the factors of the cross product in the opposite order, the resulting normal vector $\mathbf{n}$ is multiplied by -1. To remedy this ambiguity, we must always specify which of these two possible orientations of the normal vector we intend. You should always check to ensure that the normal vector is pointing in the correct direction: typical conventions are for it to be pointing "outward" or "upward".

- By plugging these expressions into the surface integral formula, we obtain explicit formulas for the outward normal flux across a surface S:
  - If S is parametrized by $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$, then the outward normal flux across S is equal to $\iint_S \mathbf{F} \cdot \left(\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right) ds\, dt$, provided that $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$ is the outward-pointing normal vector to the surface. (Conveniently, the unpleasant part of the surface-area differential cancels out the normalization in the unit normal vector.)
  - If S is defined implicitly by $f(x,y,z)=c$ and R is the projection of S in the xy-plane, then the outward normal flux across S is equal to $\iint_R \dfrac{\mathbf{F}\cdot\nabla g}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx$. Note here that the denominator term $\nabla g \cdot \mathbf{k}$ is simply the partial derivative g_z.
  - Depending on the description of the surface, either of these particular approaches (via a parametrization or as an implicit surface) may be more convenient for computing a flux integral.

- Example: Find the outward flux of the vector field $\mathbf{F} = \langle xz^2, yz^2, x^3 e^y\rangle$ through the portion of the cylinder $x^2+y^2=4$ that lies between the planes $z=-1$ and $z=1$.
  - From cylindrical coordinates, we can parametrize the cylinder as $\mathbf{r}(s,t) = \langle 2\cos t, 2\sin t, s\rangle$, where the desired portion corresponds to $-1 \le s \le 1$ and $0 \le t \le 2\pi$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial t} = \langle -2\sin t,\, 2\cos t,\, 0\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 0,0,1\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 2\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos t, 2\sin t, 0\rangle$.
  - This is indeed an outward-pointing normal vector since it is the vector pointing from $(0,0,s)$ to the point $\mathbf{r}(s,t) = (2\cos t, 2\sin t, s)$ on the surface.
  - Then $\mathbf{F} \cdot \left(\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s}\right) = \langle 2s^2\cos t, 2s^2 \sin t, (2\cos t)^3 e^{2\sin t}\rangle \cdot \langle 2\cos t,\, 2\sin t,\, 0\rangle = 4s^2\cos^2 t + 4s^2 \sin^2 t = 4s^2$.
  - The flux integral is thus $\int_0^{2\pi}\int_{-1}^{1} 4s^2\, ds\, dt = \int_0^{2\pi} \dfrac{8}{3}\, dt = \boxed{\dfrac{16\pi}{3}}$.

- Example: Find the outward flux of the vector field $\mathbf{F} = \langle x-z,\, y, x+z\rangle$ through the portion of the sphere $x^2+y^2+z^2=4$ that lies above the plane $z=1$.
  - We use the formula for flux across an implicit surface.
  - On the sphere, $z=1$ corresponds to $x^2+y^2=3$, and as z increases to 2, the value of x^2+y^2 decreases to 0. Thus the projection of the surface into the xy-plane is the region $R : x^2+y^2 \le 3$.
  - We have $\nabla g = \langle 2x, 2y, 2z\rangle$, so $\dfrac{\mathbf{F}\cdot\nabla g}{|\nabla g\cdot\mathbf{k}|} = \dfrac{2x^2-2xz+2y^2+2xz+2z^2}{2z} = \dfrac{4}{\sqrt{4-x^2-y^2}}$.

- The flux integral is therefore given by $\iint_R \dfrac{4}{\sqrt{4-x^2-y^2}}\, dy\, dx$. We will evaluate this integral using polar coordinates.
- In polar coordinates, the region is $0 \le r \le \sqrt{3}$ and $0 \le \theta \le 2\pi$, so the integral is $\int_0^{2\pi}\int_0^{\sqrt{3}} \dfrac{4}{\sqrt{4-r^2}}\, r\, dr\, d\theta$.
- Substituting $u = 4 - r^2$ in the inner integral gives $\int_0^{2\pi}\int_0^{\sqrt{3}} \dfrac{4}{\sqrt{4-r^2}}\, r\, dr\, d\theta = \int_0^{2\pi}\int_1^0 -\dfrac{2}{\sqrt{u}}\, du\, d\theta = \int_0^{2\pi} 4\, d\theta = \boxed{8\pi}$.
- Alternatively, we could have observed that for a sphere of radius ρ centered at the origin, the outward unit normal vector is $\mathbf{n} = \dfrac{1}{\rho}\langle x, y, z\rangle$.
- The desired integral is therefore $\iint_S \dfrac{1}{2}\langle x, y, z\rangle \cdot \langle x-z, y, x+z\rangle\, d\sigma = \iint_S \dfrac{1}{2}(x^2+y^2+z^2)\, d\sigma = \iint_S 2\, d\sigma$.
- This is twice the surface area of S, which we could compute (using a simpler surface integral) to be 4π, meaning that the desired flux is again $\boxed{8\pi}$.

- Example: Find the outward flux of the vector field $\mathbf{F} = \langle x, y, z\rangle$ through the sphere $x^2+y^2+z^2 = 9$.
  - Using spherical coordinates, we can parametrize the sphere as $\mathbf{r}(s,t) = \langle 3\sin s\cos t, 3\sin s\sin t, 3\cos s\rangle$ for $0 \le s \le \pi$ and $0 \le t \le 2\pi$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial t} = \langle -3\sin s\sin t, 3\sin s\cos t, 0\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 3\cos s\cos t, 3\cos s\sin t, -3\sin s\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin s\sin t & 3\sin s\cos t & 0 \\ 3\cos s\cos t & 3\cos s\sin t & -3\sin s \end{vmatrix} = \left\langle -9\sin^2 s\cos t, -9\sin^2 s\sin t, -9\sin s\cos s\right\rangle$.
  - This is not an outward-pointing normal vector, since it is $-3\sin s$ times the position vector $\mathbf{r}(s,t)$, so we must scale it by -1.
  - Then $\mathbf{F}\cdot -\left(\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s}\right) = \langle 3\sin s\cos t, 3\sin s\sin t, 3\cos s\rangle \cdot \left\langle 9\sin^2 s\cos t, 9\sin^2 s\sin t, 9\sin s\cos s\right\rangle = 27\sin^3 s\cos^2 t + 27\sin^3 s\sin^2 t + 27\sin s\cos^2 s = 27\sin s$.
  - The flux integral is thus $\int_0^{2\pi}\int_0^{\pi} 27\sin s\, ds\, dt = \int_0^{2\pi} 54\, dt = \boxed{108\pi}$.

4.4 Conservative Vector Fields, Path-Independence, and Potential Functions

- If we have a vector field $\mathbf{F}(x,y)$ and two different paths C_1 and C_2 between the same two points, we might wonder if there is any relation between the work integrals $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

- Example: For the fields $\mathbf{F}(x,y) = \langle y, x\rangle$ and $\mathbf{G}(x,y) = \left\langle y^2, x\right\rangle$ evaluate the work integrals from $(0,0)$ to $(1,1)$ along the the three different paths $C_1 : (x,y) = (t,t)$, $C_2 : (x,y) = (t^3, t^2)$, and $C_3 : (x,y) = (t^7, t^{10})$, for $0 \le t \le 1$.
  - Along C_1 we have $\mathbf{F} = \langle t, t\rangle$, $\mathbf{G} = \left\langle t^2, t\right\rangle$, $\dfrac{dx}{dt} = 1$, and $\dfrac{dy}{dt} = 1$.
  - Then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [t\cdot 1 + t\cdot 1]\, dt = \boxed{1}$, and $\int_{C_1} \mathbf{G}\cdot d\mathbf{r} = \int_0^1 \left[t^2\cdot 1 + t\cdot 1\right] dt = \boxed{\dfrac{5}{6}}$.
  - Along C_2 we have $\mathbf{F} = \left\langle t^2, t^3\right\rangle$, $\mathbf{G} = \left\langle t^4, t^3\right\rangle$, $\dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dt} = 2t$.
  - Then $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[t^2\cdot 3t^2 + t^3\cdot 2t\right] dt = \boxed{1}$, and $\int_{C_2} \mathbf{G}\cdot d\mathbf{r} = \int_0^1 \left[t^4\cdot 3t^2 + t^3\cdot 2t\right] dt = \boxed{\dfrac{29}{35}}$.
  - Along C_3 we have $\mathbf{F} = \left\langle t^{10}, t^7\right\rangle$, $\mathbf{G} = \left\langle t^{20}, t^7\right\rangle$, $\dfrac{dx}{dt} = 7t^6$, and $\dfrac{dy}{dt} = 10t^9$.
  - Then $\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[t^{10}\cdot 7t^6 + t^7\cdot 10t^9\right] dt = \boxed{1}$, and $\int_{C_3} \mathbf{G}\cdot d\mathbf{r} = \int_0^1 \left[t^{30}\cdot 7t^6 + t^7\cdot 10t^9\right] dt = \boxed{\dfrac{389}{459}}$.

- Observe that for $\mathbf{F}$, all three paths give the same value, while for $\mathbf{G}$, each path gives a different value.

- We would like to understand what about $\mathbf{F}$ in the example above seems to cause it to do the same amount of work regardless of the path we chose.

- Definition: A vector field $\mathbf{F}$ is conservative on a region R if, for any two paths C_1 and C_2 (inside R) from P_1 to P_2, it is true that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. In other words, $\mathbf{F}$ is conservative if any two paths yield the same work integral.

  - Equivalent to the above definition is the following: $\mathbf{F}$ is conservative on a region R if, for any closed curve C in R, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (A closed curve is one whose start and end points are the same.)
  - Notation: For a line integral around a closed curve, we often use the notation $\oint_C$, the circle being a suggestive example of a closed curve.
  - These two statements are equivalent because, if C_1 and C_2 are two paths from P_1 to P_2, then we can construct a closed path C by following C_1 from P_1 to P_2 and then following C_2 from P_2 back to P_1. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, and so the left-hand side is zero if and only if the right-hand side is zero.

- It turns out that we can give a simple but very useful criterion for when a vector field is conservative:

- Theorem (Fundamental Theorem of Calculus for Line Integrals): The vector field $\mathbf{F}$ is conservative on a simply-connected region R if and only if there exists a function U, called a potential function for $\mathbf{F}$, such that $\mathbf{F} = \nabla U$. If such a function U exists, then $\int_a^b \mathbf{F} \cdot d\mathbf{r} = U(b) - U(a)$ along any path from a to b.

  - Notice the similarity of the statement $\int_a^b \mathbf{F} \cdot d\mathbf{r} = U(b) - U(a)$ to the Fundamental Theorem of Calculus, which relates the integral of a derivative of a function to its values at the endpoints of a path.
  - Technical Note: The term "simply-connected" is a technical requirement needed for the proof of the theorem: intuitively, a simply-connected region consists of a single piece that does not have any "holes" in it. More rigorously, it means that the region is connected (contains only one "piece") and that if we take any closed loop in the region, we can shrink it to a point without leaving the region. The disc $x^2 + y^2 \le 4$ is simply-connected, whereas the annulus $1 \le x^2 + y^2 \le 4$ is not.
  - The full proof is not especially enlightening. We will instead show one direction of the proof.
  - Proof (Reverse Direction in 3-Space): Suppose that $\mathbf{F} = \nabla U = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle$.
  - By the (multivariable) Chain Rule, if C is the path with $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \le t \le b$, then $\frac{dU}{dt} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}$.
  - Now we can write

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\ &= \int_a^b \left[\frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt} \right] dt \\ &= \int_a^b \left[\frac{dU}{dt} \right] dt = U(\mathbf{r}(b)) - U(\mathbf{r}(a)) \end{aligned}$$

    where we used the Fundamental Theorem of Calculus for the last step.
  - Notice that this expression does not depend on C: it only involves the potential function U and the two endpoints $\mathbf{r}(b)$ and $\mathbf{r}(a)$. Hence we see that the integral is independent of the path, so $\mathbf{F}$ is conservative.

- If we can see that a vector field is conservative, then it is very easy to compute work integrals: we just need to find a potential function for the vector field.

- Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle 2x + y, x \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \langle -2\cos(\pi e^t), \tan^{-1}(t) \rangle$ from $t = 0$ to $t = 1$.

- If we try to set up the integral directly using the parametrization, it will be rather unpleasant.
- However, this vector field is conservative: it is not hard to see that for $U(x, y) = x^2 + xy$, we have $\nabla U = \langle 2x + y, x\rangle = \mathbf{F}$.
- By the Fundamental Theorem of Calculus for line integrals, the work done by the vector field is then simply the value of $U(\mathbf{r}(1)) - U(\mathbf{r}(0))$.
- Since $\mathbf{r}(1) = \langle 2, \pi/4\rangle$ and $\mathbf{r}(0) = \langle -2, 0\rangle$, the work is $U(2, \pi/4) - U(-2, 0) = \boxed{\dfrac{\pi}{2}}$.

• We would like to be able to determine easily whether a given vector field is conservative. To do this, we require a preliminary definition:

• Definition: If $\mathbf{F} = \langle P, Q, R\rangle$ then the curl of $\mathbf{F}$ is defined to be the vector field $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}, \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right\rangle = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$.

- Example: If $\mathbf{F} = \langle 3x^2y, xyz, e^{xy}\rangle$ then $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \boxed{\langle xe^{xy} - xy,\ -ye^{xy},\ yz - 3x^2\rangle}$.
- If $\mathbf{F} = \langle P, Q\rangle$ is a vector field in the plane then we define the curl of $\mathbf{F}$ to be the curl of the vector field $\langle P, Q, 0\rangle$: namely, $\left\langle 0, 0, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right\rangle$.
- Since this vector only has one nonzero component, some authors define the curl of a vector field in the plane to be the scalar quantity $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$. We will not do this: for us, the curl of a vector field will always be a new vector field.

• The curl of a vector field determines whether or not it is conservative:

• Theorem (Zero Curl Implies Conservative): A vector field on a simply-connected region in the plane or in 3-space is conservative if and only if its curl is zero. More explicitly, the vector field $\mathbf{F} = \langle P, Q\rangle$ is conservative on a simply-connected region R in the plane if and only if $P_y = Q_x$, and the vector field $\mathbf{F} = \langle P, Q, R\rangle$ is conservative on a simply-connected region D in 3-space if and only if $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.

- It is fairly easy to see why we need the equality of the derivatives of the components: if $\mathbf{F} = \langle P, Q\rangle = \nabla U$ then $P = U_x$ and $Q = U_y$, so by the equality of mixed partial derivatives, we see that $P_y = U_{xy} = U_{yx} = Q_x$.
- The three necessary equalities when $\mathbf{F} = \langle P, Q, R\rangle$ follow in the same way: if $\mathbf{F} = \nabla U$ then $P = U_x$, $Q = U_y$, and $R = U_z$, so $P_y = U_{xy} = U_{yx} = Q_x$, $P_z = U_{xz} = U_{zx} = R_x$, and $Q_z = U_{yz} = U_{zy} = R_y$.
- The converse statement (that zero curl implies the field is conservative) is more difficult, and we omit the verification.

• The two theorems give us an effective procedure for determining whether a field is conservative: we first check whether its curl is zero, and then (if it is) we can try to find a potential function by computing antiderivatives.

• Example: Determine whether $\mathbf{F}(x, y) = \langle x^2 + y, x + y^2\rangle$ is conservative, and if so, find a potential function.

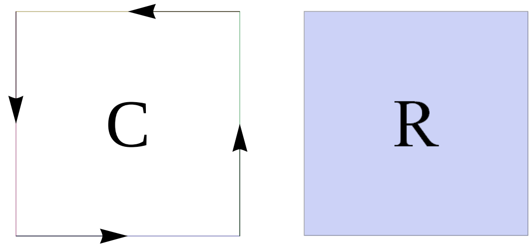
- For $\mathbf{F}$, we see $\dfrac{\partial}{\partial y}\left[x^2 + y\right] = 1 = \dfrac{\partial}{\partial x}\left[x + y^2\right]$, so the field is $\boxed{\text{conservative}}$.
- To find a potential function U with $\nabla U = \mathbf{F}$, we need to find U such that $U_x = x^2 + y$ and $U_y = x + y^2$.
- Taking the antiderivative of $U_x = x^2 + y$ with respect to x yields $U = \dfrac{1}{3}x^3 + xy + f(y)$, for some function $f(y)$.
- To find $f(y)$ we differentiate: $U_y = x + f'(y)$, so we get $f'(y) = y^2$ so $f(y) = \dfrac{1}{3}y^3$. (Plus an arbitrary constant, but we can ignore it.)
- Thus we see that a potential function for $\mathbf{F}$ is $U(x, y) = \boxed{\dfrac{1}{3}x^3 + xy + \dfrac{1}{3}y^3}$.

- Example: Determine whether $\mathbf{G}(x,y) = \langle x+y^2, x^2+y \rangle$ is conservative, and if so, find a potential function.
  - For $\mathbf{G}$, we see $\frac{\partial}{\partial y}\left[x+y^2\right] = 2y \neq 2x = \frac{\partial}{\partial x}\left[x^2+y\right]$, so the field is $\boxed{\text{not conservative}}$.
- Example: Determine whether $\mathbf{H}(x,y,z) = \langle y+2z, x+3z, 2x+3y \rangle$ is conservative, and if so, find a potential function.
  - For $\mathbf{H}$, we have $\frac{\partial}{\partial y}[y+2z] = 1 = \frac{\partial}{\partial x}[x+3z]$, $\frac{\partial}{\partial z}[y+2z] = 2 = \frac{\partial}{\partial x}[2x+3y]$, and $\frac{\partial}{\partial z}[x+3z] = 3 = \frac{\partial}{\partial y}[2x+3y]$, so the field is $\boxed{\text{conservative}}$.
  - To find a potential function U with $\nabla U = \mathbf{H}$, we need to find U such that $U_x = y+2z$, $U_y = x+3z$, and $U_z = 2x+3y$.
  - Taking the antiderivative of $U_x = y+2z$ with respect to x yields $U = xy+2xz+f(y,z)$, for some function $f(y,z)$.
  - To find $f(y,z)$ we differentiate: $x+f_y = x+3z$ and $2x+f_z = 2x+3y$, so $f_y = 3z$ and $f_z = 3y$. Repeating the process yields $f = 3yz+g(z)$, where $g'(z) = 0$.
  - Thus we see that a potential function for $\mathbf{H}$ is $U(x,y,z) = \boxed{xy+2xz+3yz}$.
- If we can find a potential function for a conservative vector field, then (as we saw above) we can use it to compute work integrals.
- Example: If $\mathbf{F} = \left\langle x^3+4x^3\sin y\,\sin z+y^2z,\ 2xyz+y+x^4\cos y\,\sin z,\ z^3+x^4\sin y\,\cos z+xy^2 \right\rangle$, find the work done by $\mathbf{F}$ on a particle that travels along the curve $C : \mathbf{r}(t) = \left\langle \sin(\pi t),\ t\sqrt{t+3},\ 2t^3+2 \right\rangle$ for $0 \le t \le 1$.
  - In theory we could compute the work integral using the parametrization of the path, but this seems quite unpleasant. Instead, we will check whether this vector field is conservative: then determining the answer only requires us to find the potential function of the field.
  - We have $P_y = 4x^3\cos y\,\sin z+2yz$ and $Q_x = 2yz+4x^3\cos y\,\sin z$ so they are equal.
  - We have $P_z = 4x^3\sin y\,\cos z+y^2$ and $R_x = 4x^3\sin y\,\cos z+y^2$ so they are also equal.
  - Finally we have $Q_z = 2xy+x^4\cos y\,\cos z$ and $R_y = 4x^3\cos y\,\cos z+2xy$, and these are also equal. Thus, the field is conservative.
  - To find a potential function U with $\mathbf{F} = \nabla U = \langle U_x, U_y, U_z \rangle$:
    * We know $U_x = x^3+4x^3\sin y\,\sin z+y^2z$ so taking the antiderivative with respect to x yields $U = \frac{1}{4}x^4+x^4\sin y\,\sin z+xy^2z+C(y,z)$.
    * We then see $U_y = x^4\cos y\,\sin z+2xyz+C_y(y,z)$ must equal $2xyz+y+x^4\cos y\,\sin z$ so we see $C_y = y$. Then taking the antiderivative with respect to y yields $C(y,z) = \frac{1}{2}y^2+D(z)$.
    * We now have $U = \frac{1}{4}x^4+x^4\sin y\,\sin z+xy^2z+\frac{1}{2}y^2+D(z)$. Then $U_z = x^4\sin y\,\cos z+xy^2+D'(z)$ must equal $z^3+x^4\sin y\,\cos z+xy^2$ so we see $D'(z) = z^3$ so we can take $D(z) = \frac{1}{4}z^4$.
  - We conclude that a potential function for $\mathbf{F}$ is $U(x,y,z) = \frac{1}{4}x^4+x^4\sin y\,\sin z+xy^2z+\frac{1}{2}y^2+\frac{1}{4}z^4$.
  - Then the desired work integral is equal to $U(0,2,4)-U(0,0,2) = \boxed{62}$.

4.5 Green's Theorem

- Green's Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus that relates a line integral of a function around a closed curve C to the double integral of a related function over the region R that is enclosed by the curve C.

- Theorem (Green's Theorem): If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then for any differentiable functions $P(x,y)$ and $Q(x,y)$, $\displaystyle\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$.
  - Here is an example of a curve C and its corresponding region R:



  - Green's Theorem, as noted above, is a generalization of the Fundamental Theorem of Calculus: both theorems show that the integral of the derivative of a function (in an appropriate sense) on a region can be computed using only the values of the function on the boundary of the region.
  - Remark: The hypotheses about the curve ("simple closed rectifiable, oriented counterclockwise") are to ensure the curve is nice enough for the theorem to hold. "Simple" means that the curve does not cross itself, "closed" means that its starting point is the same as its ending point (e.g., a circle), "rectifiable" means "piecewise-differentiable" (i.e., differentiable except at a finite number of points), and "oriented counterclockwise" means that C runs around the boundary of R in the counterclockwise direction.
  - It essentially suffices to prove Green's Theorem for rectangular regions, as more complicated regions can be built by "gluing together" simpler ones (in much the manner of a Riemann sum); overlapping boundary pieces on two rectangles sharing a side will have opposite orientations and will therefore cancel out.
  - Proof (rectangular regions): for a rectangular region $a \le x \le b$, $c \le y \le d$, we have $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$, where C_1, C_2, C_3, and C_4 are the four sides of the rectangle (with the proper orientation), and the function to be integrated on each curve is $P\,dx + Q\,dy$.
  - Setting up parametrizations shows $\int_{C_1} [P\,dx + Q\,dy] + \int_{C_3} [P\,dx + Q\,dy] = \int_a^b [P(x,c) - P(x,d)]\,dx$, and $\int_{C_2} [P\,dx + Q\,dy] + \int_{C_4} [P\,dx + Q\,dy] = \int_c^d [Q(b,y) - Q(a,y)]\,dy$.
  - For the double integral we have $\iint_R -\dfrac{\partial P}{\partial y}\,dy\,dx = \int_a^b \int_c^d -\dfrac{\partial P}{\partial y}\,dy\,dx = \int_a^b [P(x,c) - P(x,d)]\,dx$, and $\iint_R \dfrac{\partial Q}{\partial x}\,dx\,dy = \int_d^c \int_a^b \dfrac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(b,y) - Q(a,y)]\,dy$.
  - By comparing the expressions, we see that $\int_C [P\,dx + Q\,dy] = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dy\,dx$, as desired.

- Green's Theorem can be used to convert line integrals into double integrals, which can often be easier to evaluate if the curve is complicated but the region it encloses is simpler to describe.

- Example: Evaluate the integral $\oint_C 3x^2\,dx + 2xy\,dy$, where C is the counterclockwise boundary of the triangle having vertices $(0,0)$, $(1,0)$, and $(1,2)$.
  - We will evaluate the integral both as a line integral and using Green's Theorem.
  - Green's Theorem says that $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dy\,dx$, so setting $P = 3x^2$ and $Q = 2xy$ produces $\oint_C 3x^2\,dx + 2xy\,dy = \iint_R 2y\,dy\,dx$, where R is the interior of the triangle.
  - To compute the double integral, we need to describe the region R. A quick sketch shows that R is defined by the inequalities $0 \le x \le 1$ and $0 \le y \le 2 - 2x$.
  - Thus, the double integral is $\int_0^1 \int_0^{2-2x} 2y\,dy\,dx = \int_0^1 (y^2)\Big|_{y=0}^{2-2x}\,dx = \int_0^1 (2-2x)^2\,dx = \boxed{\dfrac{4}{3}}$.
  - To compute the line integral, we need to parametrize each piece of the boundary. There are three pieces.
    1. The segment from $(0,0)$ to $(1,0)$, parametrized by $x = t$, $y = 0$ for $0 \le t \le 1$. Then $dx = dt$ and $dy = 0$, so the integral here is $\int_0^1 3t^2\,dt = 1$.

2. The segment from $(1,0)$ to $(1,2)$, parametrized by $x = 1$, $y = t$ for $0 \le t \le 2$. Then $dx = 0$ and $dy = dt$, so the integral here is $\int_0^2 2t\,dt = 4$.
3. The segment from $(1,2)$ to $(0,0)$, parametrized by $x = 1-t$, $y = 2-2t$ for $0 \le t \le 1$. Then $dx = -dt$ and $dy = -2dt$, so the integral here is $\int_0^1 \left[3(1-t)^2 \cdot (-dt) + 2(1-t)(2-2t) \cdot (-2dt)\right] = \int_0^1 [-11t + 22t - 11t^2]\,dt = -\dfrac{11}{3}$.

- Thus, the value of the line integral over the entire boundary is the sum of these three, namely $1+4-\dfrac{11}{3} = \boxed{\dfrac{4}{3}}$.
- As dictated by Green's theorem, we get the same result either way. However, the double integral was quite a bit less work!

- We can use Green's Theorem to simplify the calculation of circulation and flux integrals on closed curves.
  - Specifically, we can use the theorem to give expressions for circulation and flux either as line integrals or as double integrals over a region.
  - Depending on the shape of the region and its boundary, and the nature of the field $\mathbf{F}$, either the line integral or the double integral can be easier.

- Theorem (Green's Theorem, Tangential Form): If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then the circulation around C is equal to $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T}\,ds = \iint_R (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\,dA$.
  - Recall that if $\mathbf{F} = \langle P, Q \rangle$, then curl $\mathbf{F} = \nabla \times \mathbf{F} = \left\langle 0, 0, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right\rangle$ and $(\text{curl}\,\mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$. The curl measures how much the vector field is rotating around a given point.
  - Thus, if we write everything out in terms of vector field components, the tangential form of Green's Theorem reads $\oint_C P\,dx + Q\,dy = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dy\,dx$, which is just the statement we gave above.

- Theorem (Green's Theorem, Normal Form): If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then the flux across C is equal to $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{N}\,ds = \iint_R (\text{div}\,\mathbf{F})\ dA$.
  - Here, if $\mathbf{F} = \langle P, Q \rangle$ then div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. This is called the divergence of $\mathbf{F}$ and measures how much the vector field is pushing inward or outward at the given point.
  - Explicitly, the normal form of Green's Theorem reads $\oint_C P\,dy - Q\,dx = \iint_R \left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}\right) dy\,dx$, which we can recognize as the original statement of Green's Theorem except with $-Q$ in place of P and P in place of Q.
  - There is a nice interpretation of the normal form of Green's Theorem: imagine that $\mathbf{F}$ is modeling population movement, and that C is the border of a country taking up the region R. At a city along the border C, the value $\mathbf{F} \cdot \mathbf{N}$ measures the immigration (in or out) to that city from across the border. At a city inside the country, the value div $\mathbf{F}$ measures the net immigration (into or out of) that city.
  - The normal form of Green's Theorem then says: if we add up the net immigration along the border, this equals the total population flow inside the country. (These two quantities are definitely equal, since they both tally the net immigration into the country as a whole.)

- Example: Find the outward flux through, and the (counterclockwise) circulation around, the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$, for the vector field $\mathbf{F}(x,y) = \langle x^2 - 2xy,\ y^3 - x \rangle$.
  - We could parametrize the boundary of this region and evaluate the line integrals to find the flux and circulation. However, this would be very tedious, as it requires computing four line integrals each time (one for each side of the square). We can save a lot of effort by using Green's Theorem, which applies because the boundary is a closed curve.

- For the flux, Green's Theorem says that Flux across $C = \oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} \right) dy\, dx$.
- Here, we have $P = x^2 - 2xy$ and $Q = y^3 - x$, and the region is $0 \le x \le 2$ and $0 \le y \le 2$.
- Therefore, since $\dfrac{\partial P}{\partial x} = 2x - 2y$ and $\dfrac{\partial Q}{\partial y} = 3y^2$, the flux is $\displaystyle\int_0^2 \int_0^2 \left(2x - 2y + 3y^2\right) dy\, dx = \int_0^2 \left(2xy - y^2 + y^3\right)\Big|_{y=0}^{2} dx = \int_0^2 (4x+4)\, dx = \boxed{16}$.
- Green's Theorem also says that Circulation around $C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dy\, dx$.
- Since $\dfrac{\partial Q}{\partial x} = -1$ and $\dfrac{\partial P}{\partial y} = -2x$, the circulation is $\displaystyle\int_0^2 \int_0^2 (-1 + 2x)\, dy\, dx = \int_0^2 (-2 + 4x)\, dx = \boxed{4}$.

• <u>Example</u>: For $\mathbf{F}(x, y) = \langle -x^2 y,\, xy^2 \rangle$, find the outward flux through and the (counterclockwise) circulation around the circle $x^2 + y^2 = 4$.

- We apply Green's Theorem: in this case, the region R is the region $x^2 + y^2 \le 4$.
- The flux is $\iint_R \left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} \right) dy\, dx = \iint_R (-2xy + 2xy)\, dA = \iint_R 0\, dA = \boxed{0}$.
- The circulation is $\iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dy\, dx = \iint_R (y^2 + x^2)\, dA = \int_0^{2\pi} \int_0^2 r^2 \cdot r\, dr\, d\theta = \boxed{8\pi}$, upon switching to polar coordinates.

• One of the many applications of Green's Theorem is to give various ways to compute the area of a planar region using a line integral around its boundary. Specifically, if C is the counterclockwise boundary curve of the region R (and C and R satisfy the hypotheses of Green's Theorem), then

$$\text{Area of } R = \oint_C x\, dy = \oint_C -y\, dx = \oint_C \frac{1}{2}(x\, dy - y\, dx)$$

because by Green's Theorem, each of the line integrals is equal to $\iint_R 1\, dy\, dx$, which is the area of R.

- One physical application of this idea is the construction of planimeters: they are devices used for measuring the area of a region that operate by tracing along its boundary.
- The basic principle is that the planimeter measures the amount of movement perpendicular to its measuring arm: integrating the resulting dot product around the boundary of the curve, per Green's theorem, then yields the area.

• <u>Example</u>: Compute the area enclosed by the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

- Using the third formula, we compute $A = \oint_C \dfrac{1}{2}(x\, dy - y\, dx) = \int_0^{2\pi} \dfrac{1}{2}\left[(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\right] dt = \int_0^{2\pi} \dfrac{ab}{2}\, dt = \boxed{\pi ab}$.

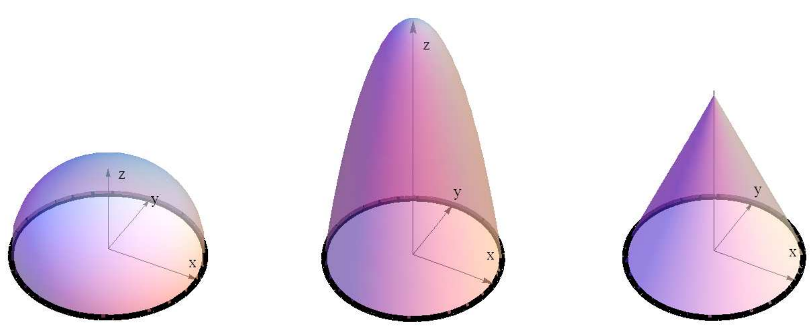
4.6 Stokes's Theorem and Gauss's Divergence Theorem

• We now discuss two generalizations of Green's theorem to 3 dimensions: these are Stokes's Theorem and Gauss's Divergence Theorem.

- As with Green's Theorem, these theorems can be used in either direction, depending on which integral is easier to set up and evaluate.
- Indeed, taken together, the Fundamental Theorem of Calculus for line integrals, Green's Theorem, Stokes's Theorem, and Gauss's Divergence Theorem collectively unify all of our notions of integration, and are all different generalizations of the Fundamental Theorem of Calculus.
- They all relate the integral of a function on the boundary of a region to the integral of a derivative on the interior of the region.
- Symbolically, their statements all read as $\displaystyle\int_{\partial R} d\omega = \int_R \omega$, where $d\omega$ represents an appropriate differential of a function ω and ∂R represents the boundary of the region R.

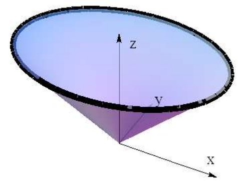
4.6.1 Stokes's Theorem

- We begin with Stokes's theorem, which is the 3-dimensional version of the tangential form of Green's theorem:
- Theorem (Stokes's Theorem): If C is a simple closed rectifiable curve in 3-space that is oriented counterclockwise around the surface S, then the circulation around C is given by $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{T}$ is the unit tangent to the curve and $\mathbf{n}$ is the unit normal to the surface.
  - Important Note: The curve C must run counterclockwise around S: in other words, when walking along C, the surface should be on its left-hand side. If one wishes to set up a problem where a curve runs clockwise around a surface, it is equivalent to traversing the curve in the opposite direction, and so the integral will be scaled by -1.
  - The hypotheses about the curve ("simple closed rectifiable, oriented counterclockwise") are the same as in Green's Theorem, and they ensure the curve is nice enough for the theorem to hold.
  - Recall $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$ if $\mathbf{F} = \langle P, Q, R \rangle$.
  - Intuitively, if we think of a vector field as modeling the flow of a fluid, the quantity $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}$ at (x, y, z) measures how much the fluid is circulating around the point (x, y, z) along the surface. Stokes's Theorem then says: we can measure how much the fluid circulates around the whole surface by measuring how much it circles around its boundary.
  - The proof of Stokes's Theorem (which we omit) is essentially the same as the proof of Green's Theorem: we can reduce to the case of showing the result for "simple" patches on the surface. Then, by parametrizing the patches explicitly, we can show Stokes's Theorem is essentially the same as the tangential form of Green's Theorem on each patch.
- Stokes's Theorem generalizes the tangential form of Green's Theorem to cover 3-dimensional closed curves and the surfaces they bound. Note that unlike in Green's Theorem, there are many possible surfaces that any given curve can bound.
  - For example, the unit circle $x^2 + y^2 = 1$, $z = 0$ in the xy-plane bounds the upper portions (i.e., where $z \geq 0$) of the sphere $x^2 + y^2 + z^2 = 1$, the paraboloid $z = 2(1 - x^2 - y^2)$, and the cone $z = 1 - \sqrt{x^2 + y^2}$, as pictured below:



- Typically, we use Stokes's Theorem when the line integral over the boundary is difficult, but there is a nicer surface available.
- Example: Find the circulation of the field $\mathbf{F}(x, y, z) = \langle y^2 z^3,\, 2xyz^3,\, 3xy^2 z^2 \rangle$ around the ellipse given by the intersection of the upper half of the ellipsoid $x^2 + 2y^2 + 2z^2 = 12$ with the cone $x^2 + 2y^2 = z^2$.

- Here is a picture of the surfaces and the ellipse:



- We could write down a parametrization for this ellipse with a little bit of effort: substituting the cone's equation into the sphere's equation gives $3z^2 = 12$ hence $z = 2$. Then using the fact that $x^2 + 2y^2 = 4$ is parametrized by $x = 2\cos(t)$ and $y = \sqrt{2}\sin(t)$ gives us a parametrization for the curve as $\mathbf{r}(t) = \langle 2\cos(t), \sqrt{2}\sin(t), 2\rangle$. The resulting circulation integral does not look so wonderful, although it is possible to evaluate it.
- Another way is to try to use Stokes's Theorem. We have two obvious surfaces to choose from (ellipsoid and cone); since the curve runs counterclockwise around the ellipsoid, we will use that.
- Stokes's Theorem tells us that Circulation around $C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma$.
- We have $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle = \langle 0, 0, 0\rangle$.
- So the curl of $\mathbf{F}$ is zero. Hence $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}$ will also be zero, so we see that the circulation is $\boxed{0}$, without even having to set up the surface integral.

- Example: Find the flux of the curl $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the surface which is the part of the sphere $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward normal.
  - We will use Stokes's Theorem. In this case, we want S to be the part of the sphere $x^2 + y^2 + z^2 = 25$ which is below the plane $z = 3$.
  - The boundary of this surface will be the intersection of the plane and the sphere: we see that the curve is the set of points $\{(x, y, z) : x^2 + y^2 = 16,\ z = 3\}$, which is a circle that we can parametrize as $\mathbf{r}(t) = \langle 4\cos(t),\ 4\sin(t),\ 3\rangle$ for $0 \le t \le 2\pi$.
  - However: the surface S lies below the curve C, not above it: so, when viewed from below (which is required because we are using the the outward normal), the curve runs clockwise around the surface.
  - In order to apply Stokes's Theorem, we need to reverse the orientation of the curve C, which we can do by interchanging the limits of integration: thus we start at $t = 2\pi$ and end at $t = 0$.
  - From Stokes's Theorem, the flux of the curl is given by the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$.
  - We have $P = 12\sin(t)$, $Q = -12\cos(t)$, and $R = e^{4\cos(t)+4\sin(t)}$, and also $dx = -4\sin(t)\, dt$, $dy = 4\cos(t)\, dt$, and $dz = 0\, dt$.
  - We get $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{0} \left[(12\sin(t)) \cdot (-4\sin(t)\, dt) + (-12\cos(t)) \cdot (4\cos(t)\, dt)) + e^{4\cos(t)+4\sin(t)} \cdot 0\, dt\right] = \int_{2\pi}^{0} -48\, dt = \boxed{96\pi}$.

4.6.2 Gauss's Divergence Theorem

- Now we discuss Gauss's Divergence Theorem, which is the 3-dimensional version of the normal form of Green's theorem:

- Theorem (Gauss's Divergence Theorem): If S is a closed, bounded, piecewise-smooth surface that fully encloses a solid region D, and $\mathbf{F}$ is a continuously differentiable vector field, then the flux across S is given by $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\operatorname{div} \mathbf{F})\ dV$, where $\mathbf{n}$ is the outward unit normal to the surface.
  - To get an idea of the setup, if S is the unit sphere $x^2 + y^2 + z^2 = 1$, then D would be the unit ball $x^2 + y^2 + z^2 \le 1$. If S consists of the 6 faces of the unit cube, then D would be the interior of the cube.

- Here, if $\mathbf{F} = \langle P, Q, R \rangle$ then div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.
- Intuitively, if we think of a vector field as modeling the flow of a fluid, the divergence measures whether there is a "source" or a "sink" at a given point (i.e., whether fluid is flowing inward toward that point, or outward from that point). The flux through a surface measures how much fluid is flowing across the surface.
- The Divergence Theorem then says that we can measure how much fluid is flowing in or out of a solid region by measuring how much fluid is flowing across its boundary.
- The proof of the Divergence Theorem (which we omit) is essentially the same as the proof of Green's Theorem: we reduce to the case of showing the result for rectangular boxes, and then parametrize the boxes explicitly.

- Typically, we want to use the Divergence Theorem to compute the flux through a closed surface, since it is usually easier to evaluate the triple integral than the surface integral.

- Example: Find the outward flux of the field $\mathbf{F}(x, y, z) = \langle x^3 - 3y, 2yz + 1, xyz \rangle$ through the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.
  - We could do this directly by computing the flux across each of the six faces of the cube. This is not the best idea, because it would require setting up six surface integrals.
  - Instead, we use the Divergence Theorem: it says Flux across $S = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\text{div } \mathbf{F})\ dV$.
  - The solid region V is defined by $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$, and div $\mathbf{F} = (3x^2) + (2z) + (xy)$.
  - Thus, the flux integral is

$$\begin{aligned} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(3x^2 + 2z + xy\right)\, dz\, dy\, dx &= \int_{-1}^{1}\int_{-1}^{1} \left(3x^2 z + z^2 + xyz\right)\Big|_{z=-1}^{1} dy\, dx \\ &= \int_{-1}^{1}\int_{-1}^{1} \left(6x^2 + 2xy\right)\, dy\, dx \\ &= \int_{-1}^{1} \left(6x^2 y + xy^2\right)\Big|_{y=-1}^{1} dx = \int_{-1}^{1} 12x^2\, dx = \boxed{8}. \end{aligned}$$

- Example: Compute the flux $\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = (x^3 + yz)\mathbf{i} + (y^3 + xz)\mathbf{j} + (z^3 + xy)\mathbf{k}$, S is the unit sphere $x^2 + y^2 + z^2 = 1$, and $\mathbf{n}$ is the outward normal.
  - We will use the Divergence Theorem. If $\mathbf{F} = \langle P, Q, R \rangle$ then $\text{div}(\mathbf{F}) = P_x + Q_y + R_z$, so here we have $\text{div}(\mathbf{F}) = 3x^2 + 3y^2 + 3z^2$.
  - The region enclosed by S is the unit ball $x^2 + y^2 + z^2 \le 1$.
  - Thus the triple integral is $\displaystyle\iiint_{x^2+y^2+z^2\le 1} \left(3x^2 + 3y^2 + 3z^2\right)\, dz\, dy\, dx$.
  - To evaluate this integral we switch to spherical coordinates: the region is bounded by the inequalities $0 \le \rho \le 1$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$, the function is $3\rho^2$, and the differential is $\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$.
  - So we obtain $\int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2 \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \dfrac{3}{5}\sin(\phi)\, d\phi\, d\theta = \int_0^{2\pi} \dfrac{6}{5}\, d\theta = \boxed{\dfrac{12\pi}{5}}$.

Well, you're at the end of my handout. Hope it was helpful.