Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													
out of	8	8	8	9	8	8	9	9	9	8	8	8	100

Math 2321 Final Exam

April 24, 2019

Instructor's name____________________ Your name____________________

Please check that you have 12 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1) Consider the surface given by the equation $z = \overbrace{3x^2y + y^2\sqrt{x}}^{f(x,y)}$ and the point $P = (1, -2, -2)$ on this surface.

(a) (4 points) Find a standard equation of the tangent plane to the surface at the point P.

$$f_x = 6xy + \frac{y^2}{2\sqrt{x}} \quad ; \quad f_y = 3x^2 + 2y\sqrt{x}$$

$$f_x(1,-2) = -10 \quad ; \quad f_y(1,-2) = -1$$

$$z = -2 + (-10)(x-1) + (-1)(y+2)$$

$$10x + y + z - 6 = 0$$

(b) (4 points) Give a unit normal vector to the tangent plane from part (a).

$$\vec{n} = (10, 1, 1)$$

$$|\vec{n}| = \sqrt{102}$$

$$\vec{u} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{102}}(10, 1, 1)$$

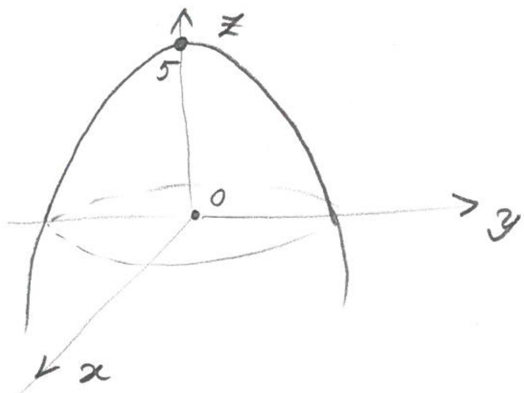
2) Consider the function $F(x, y, z) = x^2 + y^2 + z$.

(a) (4 points) Sketch the level set of F that contains the point $P = (1, 1, 3)$.

$F(1,1,3) = 5$

$x^2 + y^2 + z = 5 \Rightarrow z = 5 - x^2 - y^2$

This is the equation of a paraboloid



(b) (4 points) Give a vector equation of the line that is perpendicular to the tangent plane to the level surface from (a) through point P.

$\overrightarrow{\nabla F} = (2x, 2y, 1)$

$\overrightarrow{\nabla F}(1,1,3) = (2,2,1)$ perpendicular on the level surface
This is the direction for the line.

Line equation:

$$\begin{cases} \overrightarrow{r}(t) = (1,1,3) + t(2,2,1) \\ t \text{ real} \end{cases}$$

3) Consider the function $f(x, y) = e^{2(y-1)}\sqrt{x}$.

(a) (3 points) Starting at $(4, 1)$, in what direction does f increase most rapidly with respect to distance? Give your answer as a unit vector.

$$f_x = \frac{e^{2(y-1)}}{2\sqrt{x}} \qquad f_y = 2e^{2(y-1)}\sqrt{x}$$

$$f_x(4,1) = \frac{1}{4} \qquad f_y(4,1) = 4 \qquad \vec{\nabla} f(4,1) = \left(\frac{1}{4}, 4\right)$$

Direction: $$\frac{\vec{\nabla} f(4,1)}{\|\vec{\nabla} f(4,1)\|} = \frac{\left(\frac{1}{4}, 4\right)}{\sqrt{\frac{1}{16}+16}} = \left(\frac{1}{\sqrt{257}}, \frac{16}{\sqrt{257}}\right)$$

(b) (2 points) Determine the maximum value of the instantaneous rate of change of the function f at $(4, 1)$ with respect to distance.

$$\|\vec{\nabla} f(4,1)\| = \frac{\sqrt{257}}{4}$$

(c) (3 points) Calculate the instantaneous rate of change of f at $(4, 1)$ in the direction of $(-3, -4)$ with respect to distance.

$$\vec{u} = \frac{(-3,-4)}{\sqrt{9+16}} = \left(-\frac{3}{5}, -\frac{4}{5}\right)$$

$$D_{\vec{u}} f(4,1) = \vec{\nabla} f(4,1) \cdot \left(-\frac{3}{5}, -\frac{4}{5}\right)$$

$$= \left(\frac{1}{4}, 4\right) \cdot \left(-\frac{3}{5}, -\frac{4}{5}\right)$$

$$= -\frac{3}{20} - \frac{16}{5} = -\frac{67}{20} \overset{\text{or}}{=} -3.35$$

4) Consider the function $f(x, y) = x^3 + y^3 + 3xy + 3$.

(a) (5 points) Find all critical points of the function f.

$$\begin{cases} f_x = 3x^2 + 3y = 0 & \Rightarrow x^2 + y = 0 \\ f_y = 3y^2 + 3x = 0 & \Rightarrow y^2 + x = 0 \end{cases}$$

$\Rightarrow y = -x^2 \Rightarrow x^4 + x = 0 \Rightarrow x(x^3+1) = 0$

$\Rightarrow x = 0$ or $x^3 + 1 = 0 \Rightarrow x = -1$

$x = 0 \Rightarrow y = 0$

$x = -1 \Rightarrow y = -(-1)^2 = -1$

$\therefore$ The critical points are $(0,0)$ and $(-1,-1)$.

(b) (4 points) Classify the critical points as local minimum, local maximum, or saddle points.

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 3 \\ 3 & 6y \end{vmatrix}$$

$$D(0,0) = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0 \Rightarrow \text{saddle point at } (0,0).$$

$$D(-1,-1) = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 36 - 9 > 0$$

$$\left.\begin{matrix} D(-1,-1) > 0 \\ f_{xx}(-1,-1) = -6 < 0 \end{matrix}\right\} \Rightarrow \text{local max at } (-1,-1).$$

5) (8 points) **Using the Lagrange Multiplier Method**, find the point P on the surface $12x + 4y + 3z = 169$ closest to the origin $O = (0, 0, 0)$. (**Hint**: Minimize the square distance from the point P to the point O).

No credit is given for using a different method than the Lagrange Multiplier method.

The distance from a point (x, y, z) to $(0, 0, 0)$ is given by $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$

$f(x,y) = x^2+y^2+z^2$ is the function to be minimized

$g(x,y) = 12x+4y+3z = 169$ is the constraint fct.

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = c \end{cases} \Rightarrow \begin{cases} 2x = 12\lambda \\ 2y = 4\lambda \\ 2z = 3\lambda \\ 12x+4y+3z=169 \end{cases} \Rightarrow \begin{matrix} x = 6\lambda \\ y = 2\lambda \\ z = 1.5\lambda \end{matrix}$$

$$12 \cdot 6\lambda + 8\lambda + 4.5\lambda = 169 \Rightarrow \lambda = \frac{169}{84.5} = 2$$

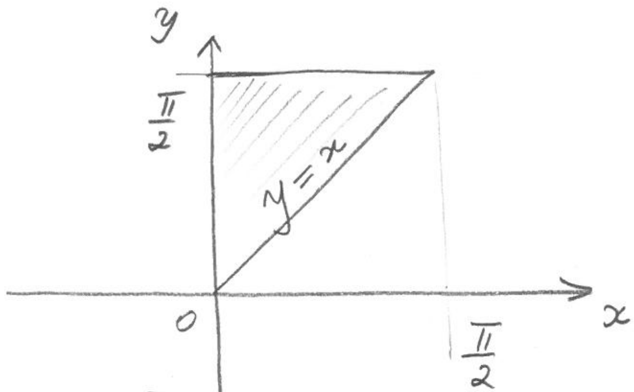
$$\Rightarrow \boxed{\begin{matrix} x = 12 \\ y = 4 \\ z = 3 \end{matrix}}$$

Check: $(12, 4, 3) \perp$ plane w eq $12x+4y+3z=169$

$$12 \cdot 12 + 4 \cdot 4 + 3 \cdot 3 = 169$$

6) (8 points) Evaluate the following integral by switching the order of integration:

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin(y)}{y}\, dy\, dx.$$



$$= \int_0^{\pi/2}\int_0^{y} \frac{\sin(y)}{y}\, dx\, dy$$

$$= \int_0^{\pi/2} y \cdot \frac{\sin(y)}{y}\, dy$$

$$= \int_0^{\pi/2} \sin(y)\, dy$$

$$= -\cos(y)\Big|_0^{\frac{\pi}{2}}$$

$$= 1.$$

$$\overbrace{\quad}^{P}\ \overbrace{\quad}^{Q}$$

7) Consider the vector field $\mathbf{F} = (6x^3y^2, 3x^4y)$.

(a) (3 points) Show that $\mathbf{F}$ is a conservative vector field.

$$Q_x = 12x^3y$$

$$P_y = 12x^3y$$

$$Qx = Py \Rightarrow \vec{F} \text{ is conservative}$$

(b) (3 points) Find a function f such that $\mathbf{F} = \nabla f$. Show work!

$$\vec{\nabla} f = \vec{F} \Rightarrow (f_x, f_y) = (6x^3y^2, 3x^4y)$$

$$\Rightarrow \begin{cases} f_x = 6x^3y^2 \\ f_y = 3x^4y \end{cases}$$

$$f = \frac{3}{2}x^4y^2 + C \text{ where } C \text{ is a constant or } f = \frac{3}{2}x^4y^2$$

(c) (3 points) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C consists of the curve $y = xe^{x^2-1} + \sin(\pi x)$ from $(0,0)$ to $(1,1)$ followed by the curve $x = \sqrt{1 + (y-1)^3}$ from $(1,1)$ to $(0,0)$.

By the Fundamental Theorem of Line Integras

$$\int_C \vec{F} \cdot d\vec{r} = f(0,0) - f(0,0) = 0$$

OR C is a closed curve and $\vec{F}$ conservative

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$$

8) (9 points) Find the surface area of the surface that is cut from the saddle-shaped surface $z = xy$ by the cylinder $x^2 + y^2 = 1$.

Set $f(x,y) = xy$

$$ds = \sqrt{f_x^2 + f_y^2 + 1}\, dA = \sqrt{y^2 + x^2 + 1}\, dA$$

$$\text{Area} = \iint_M \sqrt{y^2 + x^2 + 1}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left. \frac{(1+r^2)^{3/2}}{3/2} \right|_0^1 d\theta$$

$$= 2\pi \cdot \frac{1}{3}\left(2^{3/2} - 1\right)$$

or

$$= \frac{2\pi}{3}\left(2\sqrt{2} - 1\right).$$

9) (9 points) Let S be the solid region in the first octant ($x \geq 0,\ y \geq 0,\ z \geq 0$) between the spheres with equations $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$. Find the **mass** of S if the density of the solid region S is given by

$$\delta(x, y, z) = \frac{z^2}{(x^2 + y^2 + z^2)^2} \ \text{kg/m}^3.$$

Assume that x, y, and z are measured in meters.

Spherical coordinates $1 \leq \rho \leq 3$

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\delta = \frac{z^2}{(x^2+y^2+z^2)^2} = \frac{\rho^2 \cos^2\phi}{(\rho^2)^2} = \frac{\cos^2\phi}{\rho^2}$$

$$\text{Mass} = \iiint_S \delta \, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^3 \frac{\cos^2\phi}{\cancel{\rho^2}} \cdot \cancel{\rho^2} \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^3 \cos^2\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{\pi/2} 1 \, d\theta\right) \cdot \left(\int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi\right) \cdot \left(\int_1^3 1 \, d\rho\right)$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{3}\cos^3\phi\right)\Big|_0^{\frac{\pi}{2}} \cdot (3-1)$$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$\int \cos^2\phi \sin\phi \, d\phi$

$= \int -u^2 \, du$

$= -\frac{1}{3}u^3 + C$

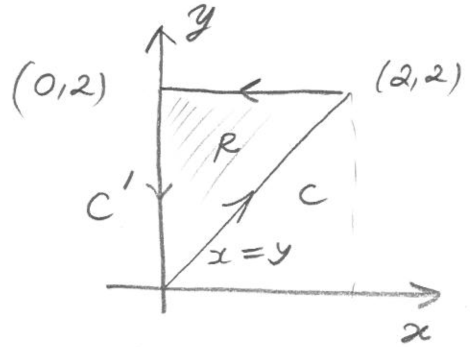
$= -\frac{1}{3}\cos^3\phi + C$

$$= \frac{\pi}{\cancel{2}} \cdot \left(-\frac{1}{3}\right)\left(\underbrace{\cos^3\tfrac{\pi}{2}}_{0} - \underbrace{\cos^3 0}_{1}\right) \cdot \cancel{2}$$

$$= \frac{\pi}{3} \text{ kg}.$$

$P \quad Q$

10) (8 points) Let $\mathbf{F} = (\sin(x^3), 2ye^{x^2})$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C consists of two line segments, which go from $(0,0)$ to $(2,2)$, and then from $(2,2)$ to $(0,2)$.



$Q_x = 4xye^{x^2}$

$P_y = 0$

$Q_x - P_y = 4xye^{x^2} \neq 0$

($\vec{F}$ is not conservative)

$$\int_C \vec{F}\cdot d\vec{r} = \underbrace{\int_{C+C'} \vec{F}\cdot d\vec{r}}_{(1)} - \underbrace{\int_{C'} \vec{F}\cdot d\vec{r}}_{(2)}$$

(1) $\overset{\text{Green's Thm}}{=} + \iint_R Q_x - P_y \, dA = \int_0^2 \int_x^2 4xye^{x^2} \, dy \, dx$ $\left(\text{or } \int_0^2 \int_0^y 4xye^{x^2} \, dx \, dy \right)$

C+C' positive oriented closed

use this to calculate

$$= \int_0^2 2y e^{x^2} \Big|_0^y \, dy = \int_0^2 2y(e^{y^2} - e^0) \, dy$$

$$= \int_0^2 2ye^{y^2} dy - \int_0^2 2y \, dy = e^{y^2}\Big|_0^2 - y^2\Big|_0^2 = e^4 - 1 - 4 + 0$$

$$= e^4 - 5$$

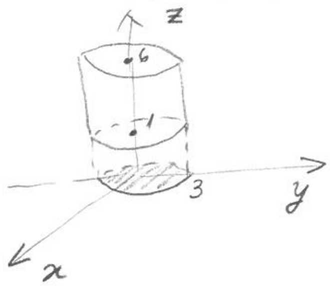
(2) $= \int_1^0 (0, 2t)\cdot(0,1)\, dt = -\int_0^1 2t \, dt = -t^2\Big|_0^1 = -1$

$\left\{ \begin{array}{l} \vec{r}(t) = (0, t) \\ t \text{ from } 1 \text{ to } 0 \end{array} \right.$

Answer $= e^4 - 5 - (-1)$

$= e^4 - 4$.

11) (8 points) Let T be the solid right circular cylinder of radius 3, centered around the z-axis, for $1 \le z \le 6$, where all lengths are in meters. Let M denote the boundary surface of the solid T; so that M consists of the cylindrical side together with disks on the top and bottom. We give M its default outward-pointing orientation. Compute the flux of the vector field $\mathbf{F}(x, y, z) = (xz, yz, xy)$ newtons through M.



$$\text{Flux} = \iint_M \vec{F}\cdot\vec{n}\, dS$$

$$= + \iiint_T \vec{\nabla}\cdot\vec{F}\, dV$$

By Divergence Theorem

$M = \partial T$

outward orientat

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xy) = z + z = 2z$$

$$\ldots = \iiint_T 2z\, dV \quad \text{using cylindrical coordinates} \quad T: \begin{cases} 0 \leqslant r \leqslant 3 \\ 0 \leqslant \theta \leqslant 2\pi \\ 1 \leqslant z \leqslant 6 \end{cases}$$

$$= \int_0^{2\pi}\int_0^3\int_1^6 2z\cdot r\, dz\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right)\cdot \int_0^3 z^2\cdot r\Big|_1^6 dr$$

$$= 2\pi \cdot 35 \cdot \int_0^3 r\, dr$$

$$= \cancel{2}\pi \cdot 35 \cdot \frac{r^2}{\cancel{2}}\Big|_0^3$$

$$= 35\cdot 9\cdot\pi = 315\pi\ N\cdot m^2$$

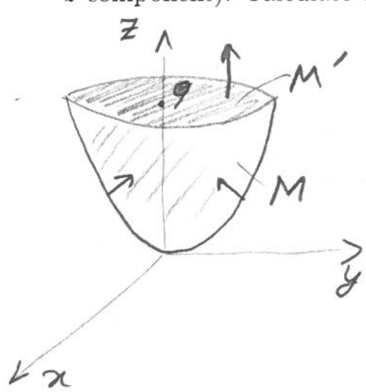
12) Let $\mathbf{F}(x, y, z) = (-x^2yz,\ xy^2z,\ xy + z^2)$.

(a) (3 points) Calculate $\vec{\nabla} \times \mathbf{F}$ = the curl of $\mathbf{F}$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2yz & xy^2z & xy+z^2 \end{vmatrix} =$$

$$(x - xy^2,\ -(y + x^2y),\ y^2z + x^2z)$$

(b) (5 points) Let M be the chopped off paraboloid given by $z = x^2 + y^2$ and $0 \leq z \leq 9$ (note that there is no closing disk at the top). Give M the upward orientation (i.e., the orientation where unit normal vectors have a positive z-component). Calculate the flux of $\vec{\nabla} \times \mathbf{F}$ through M.



By Stokes' Thm

$$\iint_M (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS =$$

$$\partial M = \partial M'$$

$$\iint_{M'} (\vec{\nabla} \times \vec{F}) \cdot (0, 0, 1)\, dA$$

upword orientation

$$= \iint_{M'} (*, *, y^2z + x^2z) \cdot (0, 0, 1)\, dA$$

$$= \iint_{M'} z(x^2+y^2)\, dA = \int_0^{2\pi}\int_0^3 9 \cdot r^2 \cdot r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right) \cdot \left(\int_0^3 9r^3\, dr\right) = 2\pi \cdot \frac{9r^4}{4}\Big|_0^3$$

$$= 2\pi \cdot \frac{9 \cdot 81}{4} = \frac{729}{2}\pi.$$

Alternate solution to part (b)

12) Let $\mathbf{F}(x, y, z) = (-x^2yz,\ xy^2z,\ xy + z^2)$.

(a) (3 points) Calculate $\vec{\nabla} \times \mathbf{F}$ = the curl of $\mathbf{F}$.

(b) (5 points) Let M be the chopped off paraboloid given by $z = x^2 + y^2$ and $0 \le z \le 9$ (note that there is no closing disk at the top). Give M the upward orientation (i.e., the orientation where unit normal vectors have a positive z-component). Calculate the flux of $\vec{\nabla} \times \mathbf{F}$ through M.

$$\iint_M (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS \ \underset{\text{By Stokes' Thm}}{=\!=\!=} \ \int_C \vec{F} \cdot dr = \ldots$$

$$\partial M = C : \begin{cases} \vec{r}(t) = (3\cos t,\ 3\sin t,\ 9) \\ 0 \le t \le 2\pi \end{cases}$$

$$\vec{r}\,'(t) = (-3\sin t,\ 3\cos t,\ 0)$$

$$\ldots = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} (-9\cos^2 t \cdot 3\sin t \cdot 9,\ 3\cos t \cdot 9\sin^2 t \cdot 9,\ *) \cdot (-3\sin t,\ 3\cos t,\ 0)$$

$$= \int_0^{2\pi} 729\cos^2 t \sin^2 t + 729 \cos^2 t \sin^2 t\, dt$$

$$= \int_0^{2\pi} 1458 \cos^2 t \sin^2 t\, dt = \int_0^{2\pi} 1458 \cdot \frac{1}{8}(1 - \cos(4t))\, dt$$

$$= \frac{1458}{8} \cdot 2\pi = \frac{729}{2}\pi$$