Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	6	8	10	10	8	10	8	10	10	10	10	100
Score:												

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Please check that you have 11 different pages.
Answers from your calculator, without supporting work, are worth zero points.

1. (6 points) Find an equation for the tangent plane to the graph of $f(x, y) = e^{2(y-1)}\sqrt{x}$ at the point $(x, y) = (4, 1)$.

$$f(x,y) = e^{2(y-1)} \cdot x^{1/2} \Rightarrow f(4,1) = 2$$

$$\Rightarrow f_x = e^{2(y-1)} \cdot \frac{1}{2x^{1/2}} \Rightarrow f_x(4,1) = \frac{1}{4}$$

$$f_y = 2e^{2(y-1)} \cdot x^{1/2} \Rightarrow f_y(4,1) = 4$$

Equation of tangent plane:

$$z = 2 + \frac{1}{4}(x-4) + 4(y-1) = \frac{x}{4} + 4y - 3$$

Ans: $\boxed{z = \frac{x}{4} + 4y - 3}$

2. An electric dipole is located at the origin in the xy-plane and generates an electrostatic potential given by

$$V(x,y) = \frac{y}{x^2 + y^2}.$$

Here V is measured in volts and x and y are measured in meters.

(a) (4 points) What is the gradient of the potential V at the point $(x,y) = (1,2)$?

$$\frac{\partial V}{\partial x} = \frac{-y}{(x^2+y^2)^2}(2x) \Rightarrow \frac{\partial V}{\partial x}(1,2) = \frac{-4}{25}$$

$$\frac{\partial V}{\partial y} = \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial V}{\partial y}(1,2) = \frac{-3}{25}$$

Gradient $\boxed{\vec{\nabla} V(1,2) = -\frac{1}{25}(4,3)}$

(b) (4 points) The level curves of V are called equipotential curves. At the point (1,2), find the direction of a vector which is tangent to the equipotential curve passing through that point. Give your answer as a unit vector with positive $\vec{i}$-component.

$\vec{\nabla} V(1,2)$ is perpendicular to $V(x,y) = 2/5$

Let $\vec{v} = (a,b)$ be perpendicular to $\vec{\nabla} V(1,2)$

$$\Rightarrow -\frac{1}{25}(4,3) \cdot (a,b) = 0$$

$$\Rightarrow 4a + 3b = 0 \Rightarrow a = -\frac{3b}{4}$$

One vector would be $(3,-4)$ by fixing $b=-4$

$$\Rightarrow \vec{v} = (3,-4) \Rightarrow \boxed{\text{Unit vector} = \left(\frac{3}{5}, \frac{-4}{5}\right)}$$

3. (10 points) Find the critical points of $f(x, y) = x^3 - 3xy + y^3$, and classify each critical point as a point where f has a local maximum value, a local minimum value, or a saddle point.

To find critical points,

$f_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y$

$f_y = -3x + 3y^2 = 0 \Rightarrow x = y^2$

Solving, we get $(y^2)^2 = y \Rightarrow y^4 = y$

$\Rightarrow y(y^3 - 1) = 0 \Rightarrow y = 0, \; y = 1$

$\Rightarrow x = 0, \; x = 1$

We have 2 critical points $(0,0)$ and $(1,1)$.

$$\left.\begin{aligned} f_{xx} &= 6x \\ f_{yy} &= 6y \\ f_{xy} &= -3 \end{aligned}\right\} D = f_{xx} f_{yy} - f_{xy}^2$$

$= 36xy - 9$

$D(0,0) = -9 \Rightarrow$ $(0,0)$ is a saddle

$D(1,1) = 27 > 0 \Rightarrow$ $(1,1)$ is a local min.

$f_{xx}(1,1) = 6 > 0$

4. (10 points) A box-shaped building with a rectangular base is to have a volume of 8000 ft^3. Suppose that annual heating and cooling costs will amount to \$2/ft^2 for its roof, front wall, and back wall, and \$4/ft^2 for the two remaining walls. Note that the floor is excluded. What dimensions of the building would minimize these annual costs?

Let x, y, z be the dimensions of the room.

$xyz = 8000 \Rightarrow z = \frac{8000}{xy}$, Cost $= 2xy + 4xz + 8yz$

Cost function $C(x, y) = 2xy + \frac{32000}{y} + \frac{64000}{x}$

Critical points $\Rightarrow$

$C_x = 2y - \frac{64000}{x^2} = 0 \Rightarrow x^2 y = 32000$ (1)

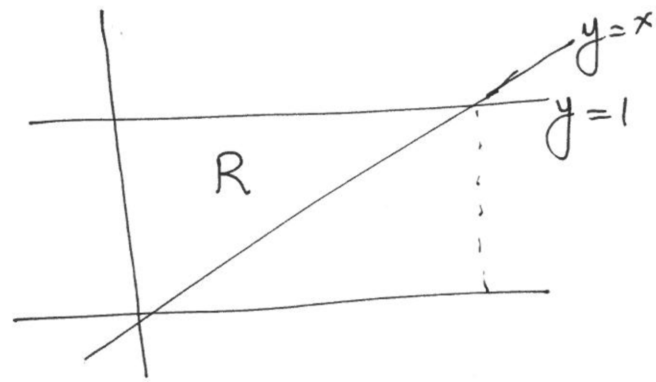
$C_y = 2x - \frac{32000}{y^2} \Rightarrow xy^2 = 16000$ (2)

(1) $\Rightarrow y = \frac{32000}{x^2}$, plugging into (2)

$x\left(\frac{32000}{x^2}\right)^2 = 16000$

$\Rightarrow x^3 = 64000 \Rightarrow \boxed{\begin{aligned} x &= 40 \\ y &= 20 \\ z &= 10 \end{aligned}}$

5. (8 points) Let R be the filled-in triangle in the first quadrant of the xy-plane which is bounded by the y-axis, the line where $y = x$, and the line where $y = 1$. Let T be the solid region in $\mathbb{R}^3$ which lies above R and below the surface where $z = 12y^2 - 12x^2 + 24$. Find the volume of the solid region T.



$$\text{Volume} = \iiint_T 1\, dV$$

$$= \iint_R \left[\int_0^{12y^2-12x^2+24} 1\, dz \right] dA$$

$$= \iint_R \left(12y^2 - 12x^2 + 24\right) dA = \int_0^1 \int_x^1 \left(12y^2 - 12x^2 + 24\right) dy\, dx$$

$$= \int_0^1 \left(28 - 12x^2 + 8x^3 - 24x\right) dx = \underline{\underline{14}}$$

Note: Can also do $\displaystyle\int_0^1 \int_0^y \left(12y^2 - 12x^2 + 24\right) dx\, dy$

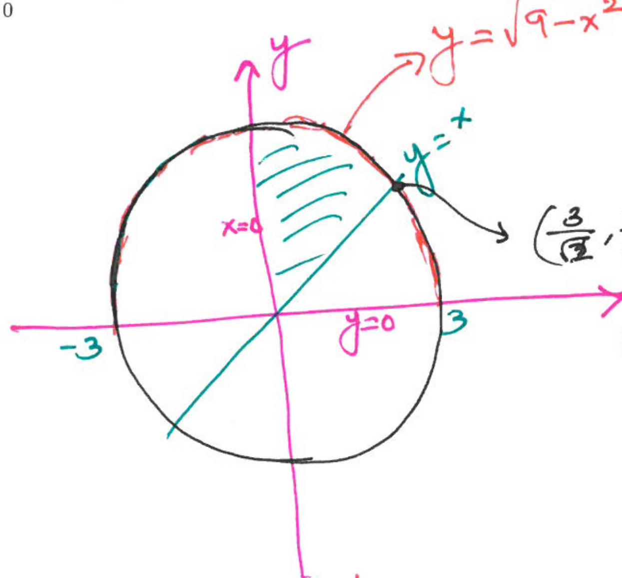
6. (10 points) Evaluate the integral

$$\int_0^{\frac{3}{\sqrt{2}}} \int_x^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} y\, dz\, dy\, dx.$$

Intersection of $y = x$ with $x^2 + y^2 = 9$

$\Longrightarrow 2x^2 = 9$

$\Longrightarrow x = \pm \frac{3}{\sqrt{2}}$



XY Projection

Cylindrical Coordinates:

$0 \le r \le 3$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$\uparrow$ $y = x$ $\quad$ $\uparrow$ $x = 0$

Triple Integral becomes

$$\int_{\pi/4}^{\pi/2} \int_0^3 \int_0^{9-r^2} \underbrace{(r\sin\theta)}_{y}\, \underbrace{dz\, r\,dr\, d\theta}_{dz\,dy\,dx}$$

$$= \int_{\pi/4}^{\pi/2} \sin\theta\, d\theta \int_0^3 (9-r^2)\, r^2\, dr = \left(\frac{1}{\sqrt{2}} - 1\right)\left(81 - \frac{243}{5}\right)$$

$$= \left(\frac{1-\sqrt{2}}{\sqrt{2}}\right) \cdot \frac{162}{5}$$

7. (8 points) Let S be the portion of the cone given by $\phi = 5\pi/6$ that is inside the sphere of radius 5. This surface can be parameterized by

$$\vec{r}\,(u,v) = \left(\frac{1}{2}u\cos v, \frac{1}{2}u\sin v, -\frac{\sqrt{3}}{2}u\right), \quad 0 \le u \le 5, \quad 0 \le v \le 2\pi.$$

Find the surface area of S.

$$\vec{r}_u = \left(\frac{1}{2}\cos v, \frac{1}{2}\sin v, -\frac{\sqrt{3}}{2}\right)$$

$$\vec{r}_v = \left(-\frac{1}{2}u\sin v, \frac{1}{2}u\cos v, 0\right)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{2}\cos v & \frac{1}{2}\sin v & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2}u\sin v & \frac{1}{2}\cos v & 0 \end{vmatrix}$$

$$= \frac{\sqrt{3}}{4}u\cos v\,\vec{i} + \frac{\sqrt{3}}{4}u\sin v\,\vec{j} + \frac{1}{4}u\,\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\underbrace{\frac{3}{16}u^2\cos^2 v + \frac{3}{16}u^2\sin^2 v}_{\frac{3}{16}u^2} + \frac{1}{16}u^2} = \frac{u}{2}$$

$$\text{Surface area} = \int_0^5\int_0^{2\pi} \frac{u}{2}\,dv\,du = \underline{\underline{\frac{25\pi}{2}}}$$

8. Consider the vector field $\mathbf{F}(x, y) = (2x^3y^4 + x, 2x^4y^3 + y^2)$ in $\mathbb{R}^2$.

(a) (2 points) Show that $\mathbf{F}$ is conservative.

$$\frac{\partial Q}{\partial x} = 8x^3y^3 = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$ defined on all of $\mathbb{R}^2$ which is s.c

$\Rightarrow$ Conservative.

(b) (4 points) Find a potential function f of $\mathbf{F}$. Show your work.

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \Rightarrow f(x,y) = \frac{x^4y^4}{2} + \frac{x^2}{2} + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2x^4y^3 + g'(y) = 2x^4y^3 + y^2$$

$$\Rightarrow g(y) = \frac{y^3}{3} (+C)$$

$$\Rightarrow f(x,y) = \frac{x^4y^4}{2} + \frac{x^2}{2} + \frac{y^3}{3} (+C)$$

(c) (4 points) Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field from part (a) and C is the half-circle of radius 2 centered at the origin and going clockwise from $(0, -2)$ to $(0, 2)$.

Using F.T of line integrals

$$\int_C \vec{F} \cdot d\vec{r} = f(0,2) - f(0,-2)$$

$$= \underline{\underline{\frac{16}{3}}}$$

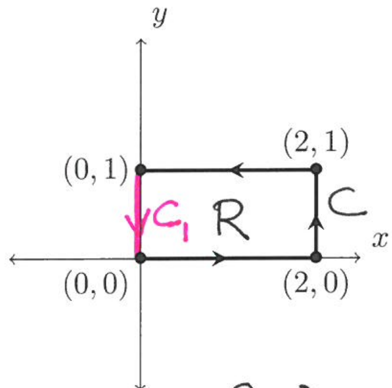
9. (10 points) Consider the vector field given by $\mathbf{F}(x,y) = (\underbrace{\sqrt{1+x^3}}_{P}, \underbrace{2xy}_{Q})$. Calculate

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where C is the curve which starts at $(0,0)$, then goes along the x-axis until it reaches $(2,0)$, then goes vertically until it reaches $(2,1)$, and the goes horizontally until it reaches $(0,1)$.

Closing side:

C_1 : $x=0$, $dx=0$



$$\int_C \vec{F}\cdot d\vec{r} = \underbrace{\int_{C+C_1} \vec{F}\cdot d\vec{r}}_{\text{Green's theorem}} - \int_{C_1} \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)\,dA - \int_{C_1} \vec{F}\cdot d\vec{r}$$

$= 0$ on C_1

$$= \iint_R 2y\,dA - \int_{y=1}^{y=0} Q\,dy$$

$$= \int_0^1\int_0^2 2y\,dx\,dy - \underbrace{\int_1^0 0\,dy}_{0} = \int_0^1 4y\,dy = 4\frac{y^2}{2}\Big]_0^1$$

$$= \underline{\underline{2}}$$

10. (10 points) Use the Divergence theorem to find the flux of the vector field

$$\mathbf{F}(x,y,z) = (0,0,z^3/3)$$

across the sphere of radius 1 centered at the origin oriented by the outward pointing normal.

Use Divergence Theorem:

$$\iint_M \vec{F}\cdot\vec{n}\,dS = \iiint_S (\vec{\nabla}\cdot\vec{F})\,dV = \iiint_S z^2\,dV$$

Method 1: Use spherical coordinates: $z = \rho\cos\phi$

$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$$\Rightarrow \int_0^{\pi}\int_0^{2\pi}\int_0^1 (\rho^2\cos^2\phi)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi = 2\pi \underbrace{\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi}_{\underbrace{\left.-\frac{\cos^3\phi}{3}\right]_0^{\pi}}_{2/3}} \underbrace{\int_0^1 \rho^4\,d\rho}_{1/5}$$

$$\boxed{= \frac{4\pi}{15}}$$

Method 2: Use cylindrical coordinates

$$\int_0^{2\pi}\int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \underbrace{z^2\,dz}_{\frac{z^3}{3}}\,r\,dr\,d\theta = \frac{2\pi}{3}\cdot 2\int_0^1 (\underbrace{1-r^2}_{})^{3/2}\,r\,dr$$

$u = 1-r^2 \Rightarrow du = -2r\,dr$

$\Rightarrow r\,dr = -\frac{du}{2}$

$$\int u^{3/2}\left(-\frac{du}{2}\right)$$

$$= -\frac{1}{2}\cdot\frac{2}{5}u^{5/2}$$

$$= -\frac{1}{5}(1-r^2)^{5/2}\Big]_0^1$$

$$\boxed{= \frac{4\pi}{15}}$$

11. Let M be the portion of the graph of $z = 1-x^2-y^2$ that lies above the xy-plane ($z \geq 0$), oriented upward. Let

$$\mathbf{F}(x, y, z) = (x^2+y^2+y+z^2, 3z, 5-x^2-y^2).$$

(a) (3 points) Let ∂M be the boundary of M. Write a parameterization of ∂M.

∂M is the unit circle in xy-plane

$\Rightarrow \vec{r}(t) = (\cos t, \sin t, 0)$

(b) (7 points) Use Stokes' theorem to compute

$$\iint_M \left(\vec{\nabla} \times \mathbf{F}\right) \cdot \vec{n}\, dS.$$

(Hint: Depending on how you solve the problem, the following fact may (or may not) be useful: $\int_0^{2\pi} (\sin x)^2\, dx = \pi$.) Be careful to explain each step in your calculation.

Method 1: $\displaystyle\iint_M (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \int_{\partial M} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$\displaystyle = \int_0^{2\pi} (1+\sin t, 0, 4) \cdot (-\sin t, \cos t, 0)\, dt$

$\displaystyle = \int_0^{2\pi} (-\sin t - \sin^2 t)\, dt = \underline{\underline{-\pi}}$

Method 2: $\displaystyle\iint_M (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \iint_{M_1} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$ where M_1 is the unit disk in XY plane, oriented upward.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2+y+z^2 & 3z & 5-x^2-y^2 \end{vmatrix}$$

$$= \left(-2y-3,\quad 2x+2z,\quad -2y-1\right)$$

$$\vec{n} = (0, 0, 1)$$

$$\iint_{M_1} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \iint_{M_1} (-2y-1)\, dA$$

$$= \int_0^{2\pi} \int_0^1 \left(-2\sin\theta - 1\right) r\,dr\,d\theta$$

$$= -\int_0^{2\pi} (2\sin\theta + 1)\, d\theta \int_0^1 r\,dr$$

$$= -2\pi \cdot \frac{1}{2} \boxed{= -\pi}$$