Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													
out of	10	8	8	8	8	8	8	8	8	8	8	10	100

Math 2321 Final Exam

December 13, 2016

Instructor's name____________________ Your name____________________

Please check that you have 10 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1) Consider the function $f(x,y,z) = x^4 + y^4 - 4x^2y^2e^z$.

a) (3 points) Find the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ of the function f at the point $(1,1,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,0)} = \left.(4x^3 - 8xy^2e^z)\right|_{(1,1,0)} = 4 - 8 = -4.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,0)} = \left.(4y^3 - 8x^2ye^z)\right|_{(1,1,0)} = 4 - 8 = -4.$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,0)} = \left.(-4x^2y^2e^z)\right|_{(1,1,0)} = -4.$$

b) (5 points) Find the linearization $L(x,y,z)$ of $f(x,y,z)$ at $(1,1,0)$.

$$f(1,1,0) = 1 + 1 - 4 = -2.$$

$$L(x,y,z) = -2 - 4(x-1) - 4(y-1) - 4(z-0).$$

c) (2 points) Use the linearization of f at $(1,1,0)$ to estimate the value of f at $(0.9, 1.1, 0.1)$. (The "exact" value of $f(0.9, 1.1, 0.1)$ from your calculator is worth zero points.)

$$f(0.9, 1.1, 0.1) \approx L(0.9, 1.1, 0.1) =$$

$$-2 - 4(-0.1) - 4(0.1) - 4(0.1)$$

$$= -2 + 0.4 - 0.4 - 0.4 = -2.4.$$

2) (8 points) Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 8x^2 + 2y^2 + 3z^2$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature decrease most rapidly when at the point $(1, 1, 2)$? What is the minimum rate of change of the temperature (with respect to distance) when at the point $(1, 1, 2)$?

$$\vec{\nabla} T(1,1,2) = (16x, 4y, 6z)\Big|_{(1,1,2)} = (16, 4, 12).$$

$$|\vec{\nabla} T(1,1,2)| = |4(4,1,3)| = 4\sqrt{16+1+9}.$$

Direction of most rapid decrease $= -\dfrac{(4,1,3)}{\sqrt{26}} \left(= -\dfrac{\vec{\nabla} T(1,1,2)}{|\vec{\nabla} T(1,1,2)|} \right).$

Min. rate of change $= -|\vec{\nabla} T(1,1,2)| = -4\sqrt{26}\ {}^\circ C/m.$

3) Consider the surface M in $\mathbb{R}^3$ where $3x^2 - y^2 + z^2 = 3$.

a) (5 points) Find an equation for the tangent plane to the surface at the point $(0, 1, 2)$.

$F(x,y,z) = 3x^2 - y^2 + z^2.$

normal vector $= \underbrace{\vec{\nabla} F(0,1,2) = (6x, -2y, 2z)\Big|_{(0,1,2)}} = (0,-2,4).$

Tangent plane: $0(x-0) - 2(y-1) + 4(z-2) = 0.$

$-2y + 2 + 4z - 8 = 0. \quad -2y + 4z - 6 = 0.$

$-y + 2z - 3 = 0.$

b) (3 points) Give a vector equation for the line which passes through the point $(0, 1, 2)$ and is normal to the surface M at $(0, 1, 2)$.

$$(x,y,z) = (0,1,2) + t(0,-2,4).$$

4) (8 points) Find the critical points of $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$ and determine what type of critical point each of them is, i.e., classify each one as a point where f has a local maximum value, a local minimum value, or a saddle point.

+1

$f_x = 12x - 6x^2 + 6y = 0.$ $\quad 2x - x^2 + y = 0.$

$f_y = 6y + 6x = 0.$ $\quad y + x = 0.$ $\quad y = -x.$

$2x - x^2 - x = 0,$ $\quad x - x^2 = 0,$ $\quad x(1-x) = 0.$

$x = 0$ or $x = 1$

$\downarrow \qquad \downarrow$

$y = 0.$ $\quad 2 - 1 + y = 0.$ $\quad y = -1.$

Critical points: $(0,0)$ and $(1,-1)$. +2

$f_{xx} = 12 - 12x.$ $\quad f_{yy} = 6.$ $\quad f_{xy} = 6.$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12(1-x) & 6 \\ 6 & 6 \end{vmatrix} = 72(1-x) - 36$$

$$= 36[2(1-x) - 1].$$

+1

At $(0,0)$:

$D = 36 > 0$, $f_{xx} = 12 > 0.$

Local min. at $(0,0)$. +2

At $(1,-1)$:

$D = -36 < 0.$

Saddle point at $(1,-1)$. +2

5) You are designing a web site for calculating the surface area of domes (arched roofs) for buildings. You decide to model a parabolic dome using the equation $z = h - b(x^2 + y^2)$ for the dome with base in the xy-plane; here, h and b are positive constants.

a) (2 points) Give a parameterization of the dome, being careful to say what the domain of the parameterization is.

$\underline{r}(x,y) = (x, y, h - b(x^2+y^2))$, where (x,y) is in the disk of radius $\sqrt{\frac{h}{b}}$, centered at the origin

OR

$\underline{r}(u,v) = (u\cos v, u\sin v, h - bu^2)$,

$0 \le u \le \sqrt{\frac{h}{b}}$,

$0 \le v \le 2\pi$.

b) (6 points) Find a formula for the surface area of the dome in terms of b and h.

$f(x,y) = h - b(x^2+y^2)$.

~~Area~~

$dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA$

$= \sqrt{(-2bx)^2 + (-2by)^2 + 1}\, dA$

$= \sqrt{4b^2(x^2+y^2)+1}\, dA$.

$\text{Area} = \iint_D \sqrt{4b^2r^2+1}\,(r\,dr\,d\theta)$

$= \int_0^{2\pi}\int_0^{\sqrt{h/b}} (4b^2r^2+1)^{\frac{1}{2}}\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_{1}^{4bh+1} u^{\frac{1}{2}} \frac{1}{8b^2}\,du\,d\theta$

Let $u = 4b^2r^2+1$,
$du = 8b^2 r\,dr$.
$\frac{1}{8b^2}du = r\,dr$.

$= \frac{2\pi}{8b^2}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_{u=1}^{u=4bh+1}$

$= \frac{4\pi}{24b^2}(4bh+1)^{3/2} - \frac{4\pi}{24b^2} =$

$\boxed{\frac{\pi}{6b^2}\left[(4bh+1)^{3/2} - 1\right]}$.

OR

$\underline{r}_u = (\cos v, \sin v, -2bu)$.

$\underline{r}_v = (-u\sin v, u\cos v, 0)$.

$\underline{r}_u \times \underline{r}_v = (2bu^2\cos v, 2bu^2\sin v, u\cos^2 v + u\sin^2 v)$

$= (2bu^2\cos v, 2bu^2\sin v, u)$.

$dS = |\underline{r}_u \times \underline{r}_v|\,du\,dv =$

$\sqrt{4b^2u^4\cos^2 v + 4b^2u^4\sin^2 v + u^2}\,du\,dv$

$= u\sqrt{4b^2u^2+1}\,du\,dv$.

since $u \ge 0$

$\text{Area} = \int_0^{2\pi}\int_0^{\sqrt{h/b}} u\sqrt{4b^2u^2+1}\,du\,dv$

6) (8 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 2x + 6y + 10z$ subject to the constraint $\underbrace{x^2 + y^2 + z^2}_{g} = 35$, and also give the points at which the maximum and minimum values occur.

$\vec{\nabla} f = \lambda \vec{\nabla} g$. $(2, 6, 10) = \lambda (2x, 2y, 2z)$.

λ Can't be 0.

Solve: $x^2 + y^2 + z^2 = 35$.

$2 = 2\lambda x$. $6 = 2\lambda y$. $10 = 2\lambda z$.

$x = \frac{2}{2\lambda} = \frac{1}{\lambda}$. $y = \frac{6}{2\lambda} = \frac{3}{\lambda}$. $z = \frac{10}{2\lambda} = \frac{5}{\lambda}$.

$\left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 = 35$.

$\frac{1}{\lambda^2} + \frac{9}{\lambda^2} + \frac{25}{\lambda^2} = 35$. $\frac{35}{\lambda^2} = 35$. $\lambda^2 = 1$. $\boxed{\lambda = \pm 1.}$

$\lambda = 1$:

$(x, y, z) = (1, 3, 5)$.

$f(1, 3, 5) = 2 + 18 + 50$
$= 70$.

Max. value at $(1, 3, 5)$

$\lambda = -1$:

$(x, y, z) = (-1, -3, -5)$.

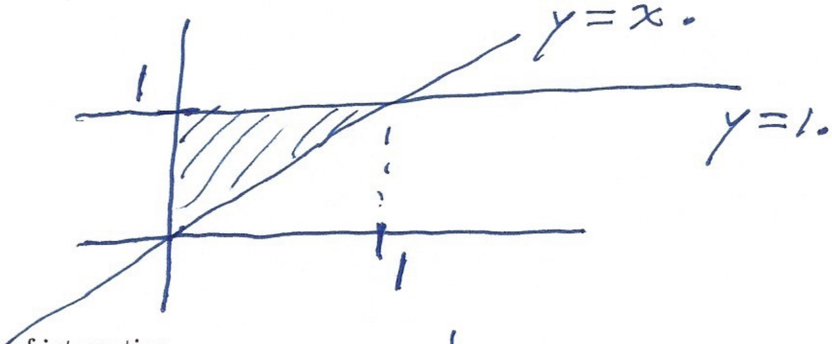
$f(-1, -3, -5) = -70$.

Min. value at $(-1, -3, -5)$.

7) Consider the iterated integral $\int_0^1 \int_x^1 e^{x/y}\,dy\,dx$.

a) (3 points) Sketch the region of integration.

$0 \le x \le 1. \quad x \le y \le 1.$



b) (5 points) Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx = \int_0^1 \int_0^y e^{x/y}\,dx\,dy = \int_0^1 \frac{e^{x/y}}{1/y}\Big|_{x=0}^{x=y} dy$$

$$= \int_0^1 (ey - y)\,dy = \int_0^1 (e-1)y\,dy =$$

$$(e-1)\frac{y^2}{2}\Big|_0^1 = \frac{1}{2}(e-1).$$

8) (8 points) A region R in $\mathbb{R}^2$ is given by $1 \le x^2 + y^2 \le 4$. Sketch R and evaluate the double integral $\int\int_R e^{x^2+y^2}\,dA$.

$$\int_0^{2\pi} \int_1^2 e^{r^2}\, r\,dr\,d\theta =$$

Let $u = r^2$.
$du = 2r\,dr$.
$\frac{1}{2}du = r\,dr$.

$$\int_0^{2\pi} \int_1^4 e^u \left(\frac{1}{2}du\right) d\theta =$$

$$\int_0^{2\pi} \left(\frac{1}{2}e^u \Big|_{u=1}^{u=4}\right) d\theta = \int_0^{2\pi} \frac{1}{2}(e^4 - e)\,d\theta$$

$$= 2\pi \cdot \frac{1}{2}(e^4 - e) = \pi(e^4 - e).$$

9) (8 points) Determine the mass of the $\overbrace{\text{solid region}}^{T}$ that lies above the xy-plane, below the half-cone where $z = 2\sqrt{x^2+y^2}$, outside of the sphere of radius 1 and inside the sphere of radius 2, both centered at the origin, if the density is given by $\delta(x,y,z) = \frac{1}{x^2+y^2+z^2}$ kg/m^3. All lengths are measured in meters.

$z = 2\sqrt{x^2+y^2}$. $z = 2r$. $\rho\cos\phi = 2\rho\sin\phi$.

$\tan\phi = \frac{1}{2}$. $\phi = \underbrace{\tan^{-1}\left(\frac{1}{2}\right)}_{\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)} \approx 0.4636.$

$$\text{Mass} = \iiint_T \delta \, dV =$$

$$\int_0^{2\pi}\int_{\cos^{-1}(2/\sqrt{5})}^{\frac{\pi}{2}}\int_1^2 \frac{1}{\rho^2}\rho^2\sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi}\int_{\cos^{-1}(2/\sqrt{5})}^{\pi/2} \rho\sin\phi \Big|_{\rho=1}^{2} d\phi \, d\theta = \int_0^{2\pi}\int_{\cos^{-1}(2/\sqrt{5})}^{\pi/2} \sin\phi \, d\phi \, d\theta$$

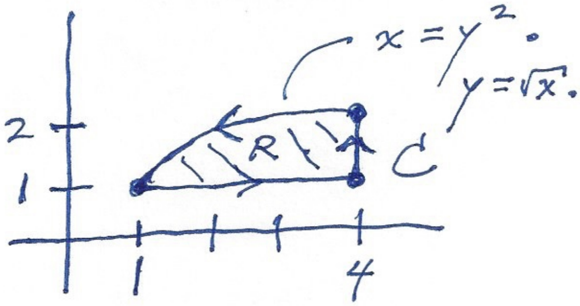
$$= \int_0^{2\pi} -\cos\phi \Big|_{\phi=\cos^{-1}(2/\sqrt{5})}^{\phi=\frac{\pi}{2}} d\theta =$$

$$\int_0^{2\pi} 0 - -\frac{2}{\sqrt{5}} \, d\theta = \frac{2}{\sqrt{5}}\theta \Big|_0^{2\pi} = \frac{4\pi}{\sqrt{5}} \text{ kg}.$$

$$\approx 5.6199.$$

$$\overset{P}{}\quad\overset{Q}{}$$

10) (8 points) Find the work done by the force field $\mathbf{F} = \left(\ln y, x\left(y+\frac{1}{y}\right)\right)$ Newtons, where x and y are in meters, on a particle that moves from $(1,1)$ along a horizontal line to $(4,1)$, then from $(4,1)$ to $(4,2)$ along a vertical line, and finally moves from $(4,2)$ back to $(1,1)$ along the curve $x = y^2$.



$$Q_x - P_y = y + \frac{1}{y} - \left(\frac{1}{y}\right) = y.$$

Green's Theorem:

$$\text{Work} = \int_C \underline{F}\cdot d\underline{r} = \iint_R (Q_x - P_y)\,dA =$$

$$\int_1^4\int_1^{\sqrt{x}} y\,dy\,dx =$$

$$\int_1^4 \frac{y^2}{2}\Big|_{y=1}^{y=\sqrt{x}} dx =$$

$$\int_1^4 \frac{x}{2} - \frac{1}{2}\,dx =$$

$$\frac{1}{2}\int_1^4 (x-1)\,dx =$$

$$\frac{1}{2}\left(\frac{x^2}{2} - x\right)\Big|_1^4 =$$

$$\frac{1}{2}\left((8-4) - \left(\frac{1}{2}-1\right)\right) =$$

$$\frac{1}{2}\left(4 - -\frac{1}{2}\right) = \frac{9}{4} \text{ joules.}$$

OR

$$\int_1^2\int_{y^2}^4 y\,dx\,dy =$$

$$\int_1^2 xy\Big|_{x=y^2}^{x=4} dy =$$

$$\int_1^2 (4y - y^3)\,dy =$$

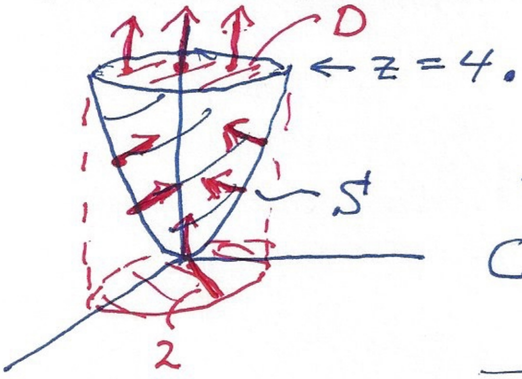
$$2y^2 - \frac{y^4}{4}\Big|_1^2 =$$

$$(8-4) - \left(2 - \frac{1}{4}\right) =$$

$$4 - 2 + \frac{1}{4} = 2 + \frac{1}{4} =$$

$$\frac{9}{4} \text{ joules.}$$

11) (8 points) Evaluate the flux integral $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = (xy^2 - 1, x^2y, 2z)$ and S is the part of the surface given by $z = x^2 + y^2$ that is above the disk of radius 2, centered at the origin, in the xy-plane. The surface is oriented upward (i.e., oriented by the upward normal).



Let $\hat{S} = -S'$, so that $\hat{S} \cup D$ is oriented outward from enclosed solid T.

Calculate $\iint_{\hat{S}} \underline{F}\cdot\underline{n}\,dS'$, then negate.

$$\iint_{\hat{S}} \underline{F}\cdot\underline{n}\,dS' = \iint_{\hat{S}\cup D} \underline{F}\cdot\underline{n}\,dS' - \iint_D \underline{F}\cdot\underline{n}\,dS' =$$

$$\iiint_T (\vec{\nabla}\cdot\underline{F})\,dV - \iint_D (*,*,8)\cdot(0,0,1)\,dA =$$

$$\iiint_T (y^2+x^2+2)\,dV - \iint_D 8\,dA =$$

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 (r^2+2)\,r\,dz\,dr\,d\theta - 8\left(\begin{matrix}\text{area}\\ \text{of } D\end{matrix}\right)$$

$$= \int_0^{2\pi}\int_0^2 (r^3+2r)(4-r^2)\,dr\,d\theta - 8\pi(2)^2 =$$

$$\int_0^{2\pi}\int_0^2 (\underbrace{4r^3 - r^5 + 8r - 2r^3}_{2r^3})\,dr\,d\theta - 32\pi =$$

$$\int_0^{2\pi} \left(\frac{r^4}{2} - \frac{r^6}{6} + 4r^2\right)\Big|_0^2\,d\theta - 32\pi = \frac{80\pi}{3} - \frac{96\pi}{3}$$

$$= -\frac{16\pi}{3}.$$

$$\boxed{\text{So, } \iint_{S'} \underline{F}\cdot\underline{n}\,dS' = \frac{16\pi}{3}.}$$

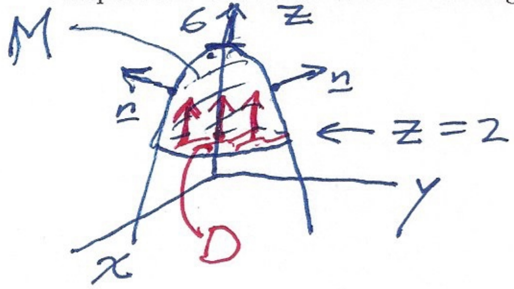
12) Let $\mathbf{F}(x, y, z) = (x^2 \sin z, -x + z^3, e^{xy})$ be a vector field.

a) (4 points) Compute the curl, $\vec{\nabla} \times \mathbf{F}$, of $\mathbf{F}$.

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 \sin z & -x + z^3 & e^{xy} \end{vmatrix} =$$

$$(x e^{xy} - 3z^2, -(y e^{xy} - x^2 \cos z), -1).$$

b) (6 points) Let M be the portion of the paraboloid $z = 6 - x^2 - y^2$ which sits above the plane $z = 2$, oriented upward. Compute the flux of the curl of $\mathbf{F}$ through M.



$$\iint_M (\vec{\nabla} \times \underline{F}) \cdot \underline{n} \, dS$$

$$= \iint_D (\vec{\nabla} \times \underline{F}) \cdot (0, 0, 1) \, dA =$$

$$\iint_D (*, *, -1) \cdot (0, 0, 1) \, dA = \iint_D -1 \, dA$$

$$= \underbrace{-(\text{area of } D)}_{\text{Find radius of } D.} = -\pi(2)^2 = -4\pi.$$

~~Re~~ Find radius of D.

$2 = 6 - x^2 - y^2$,

$x^2 + y^2 = 4$.

$r = 2$.