Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

Math 2321 Final Exam

December 12, 2013

Instructor's name__________________ Your name__________________

Please check that you have 9 different pages.

Answers from your calculator, without supporting work, are worth zero points.

Rangachev

1) A charge distribution on a plane is creating an electric field. The electrical potential $P(x, y)$ measures the potential energy of a unit point charge due to its position in the field. The function is given by $P(x, y) = \frac{2}{\sqrt{(x+2)^2+(y-1)^2}}$.

a) (4 points) Find the gradient vector of the potential at $(1, 5)$.

$$P = 2\left[(x+2)^2 + (y-1)^2\right]^{-\frac{1}{2}}.$$

$$\frac{\partial P}{\partial x} = 2\left(-\frac{1}{2}\right)\left[(x+2)^2+(y-1)^2\right]^{-\frac{3}{2}} \cdot 2(x+2) = \frac{-2(x+2)}{\left[(x+2)^2+(y-1)^2\right]^{\frac{3}{2}}}. \quad \left.\frac{\partial P}{\partial x}\right|_{(1,5)} = \frac{-6}{125}.$$

$$\frac{\partial P}{\partial y} = 2\left(-\frac{1}{2}\right)\left[(x+2)^2+(y-1)^2\right]^{-\frac{3}{2}} \cdot 2(y-1) = \frac{-2(y-1)}{\left[(x+2)^2+(y-1)^2\right]^{\frac{3}{2}}}. \quad \left.\frac{\partial P}{\partial y}\right|_{(1,5)} = \frac{-8}{125}.$$

3 pts.

$$\vec{\nabla} P(1,5) = \left(-\frac{6}{125}, -\frac{8}{125}\right) = -\frac{2}{125}(3,4)$$ 1 pt.

b) (4 points) An equipotential line is a curve on our plate along which the potential is constant. What is an equation for the tangent line of the equipotential line passing through $(1, 5)$?

$$3(x-1) + 4(y-5) = 0.$$

or

$$\underbrace{-\frac{6}{125}(x-1) - \frac{8}{125}(y-5)}_{\text{2 pts.}} = 0.$$

1 pt. 1 pt.

Gaffney

2) (8 points) Find the critical points of $f(x,y) = x^3 + 8y^3 - 6xy$, verify that each critical point is non-degenerate, and determine what type of critical point it is.

$$\left.\begin{aligned} f_x &= 3x^2 - 6y = 0. \\ f_y &= 24y^2 - 6x = 0. \end{aligned}\right\}\ 1\text{pt.} \qquad \left.\begin{aligned} x^2 &= 2y. \\ 4y^2 &= x. \end{aligned}\right\}$$

$(4y^2)^2 = 2y.$ $16y^4 = 2y.$

$16y^4 - 2y = 0.$

2 pts. $2y(8y^3 - 1) = 0.$

$\left\{\begin{matrix} y = 0 \\ x = 0 \end{matrix}\right.$ or $\begin{matrix} y = \frac{1}{2}. \\ x = 1. \end{matrix}$

Two critical points: $(x,y) = (0,0)$ and $(x,y) = (1, \frac{1}{2})$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -6 \\ -6 & 48y \end{vmatrix} = 6(48xy) - 36 = 36(8xy - 1).$$

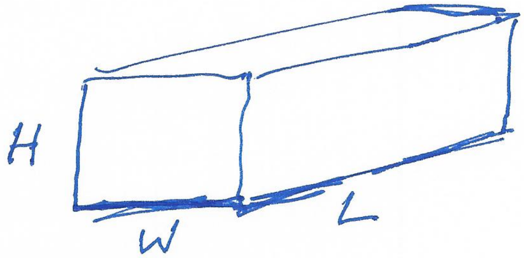
2 pts.

At $(0,0)$, $D = -36 < 0$. Saddle point. 1 pt.

At $(1, \frac{1}{2})$, $D = (36)(3) > 0$. $f_{xx} = 6 > 0$. Local min. 2 pts.

Rainho

3) (10 points) Suppose that a cardboard box is to be constructed with **no** top and a volume of 4000 cubic inches. Suppose that the cardboard for the bottom costs 5 cents per square inch, while the cardboard for the sides costs 1 cent per square inch. Find the dimensions of the box which minimize the cost of the cardboard required.



$4000 = LWH.$ ← 1 pt.

2 pts.

$C = \underset{\text{in } ¢}{\text{cost}} = 5LW + 1\cdot 2LH + 1\cdot 2WH.$

Minimize C. Solve for H : $H = \dfrac{4000}{LW}$.

(or for L or W)

2 pts.

$$C = 5LW + 2L\left(\frac{4000}{LW}\right) + 2W\left(\frac{4000}{LW}\right).$$

$$C = 5LW + 8000W^{-1} + 8000L^{-1}.$$

Find critical point(s).

$$\left.\begin{aligned} \frac{\partial C}{\partial L} &= 5W - 8000L^{-2} = 0. \quad W = \frac{1600}{L^2}. \\ \frac{\partial C}{\partial W} &= 5L - 8000W^{-2} = 0. \quad L = \frac{1600}{W^2}. \end{aligned}\right\} \text{2 pts.}$$

$$W = \frac{1600}{\left(\frac{1600}{W^2}\right)^2} = \frac{W^4}{1600}.$$

$\cancel{W = 0}$ ← $LWH = 4000$, or $W^3 = 1600.$

$W = (1600)^{1/3}$ inches. $L = \dfrac{1600}{(1600)^{2/3}} = (1600)^{1/3}$ in.

$H = \dfrac{4000}{(1600)^{2/3}}$ in. 3 pts.

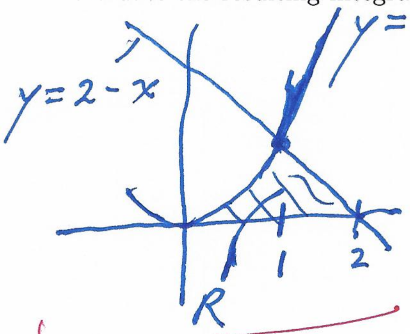
or $W = L = 4(5)^{2/3}$ in.
$H = 10(5)^{2/3}$ in.

or
$W = L \approx 11.696$ in.
$H \approx 29.240$ in.

$y=0$ to $y=x^2$, $0 \le x \le 1$ $\qquad$ $y=0$ to $y=2-x$, $1 \le x \le 2$

Hepler

4) (8 points) For the following sum of integrals, $\int_0^1 \int_0^{x^2} y\,dy\,dx + \int_1^2 \int_0^{2-x} y\,dy\,dx$, reverse the order of integration and evaluate the resulting integral using this new order.



3 pts.
If no sketch, absorb points into limits of int.

$$\iint_R y\,dA = \underbrace{\int_0^1 \int_{\sqrt{y}}^{2-y} y\,dx\,dy}_{2\text{ pts.}} =$$

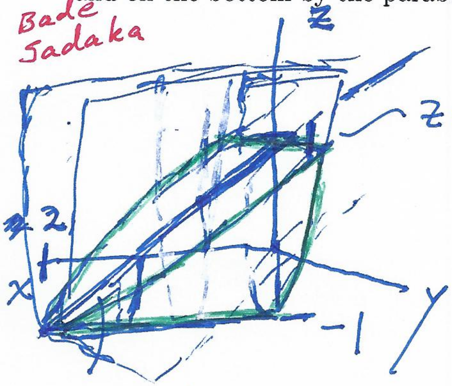
$$\int_0^1 yx \Big|_{x=y^{\frac12}}^{x=2-y} dy = \quad 1\text{ pts.}$$

$$\int_0^1 y(2-y) - y^{\frac32}\,dy = \int_0^1 2y - y^2 - y^{\frac32}\,dy$$

$$= y^2 - \frac{y^3}{3} - \frac{y^{\frac52}}{\frac52}\Big|_0^1 = 1 - \frac13 - \frac25 = \frac{4}{15} \quad 2\text{ pts.}$$

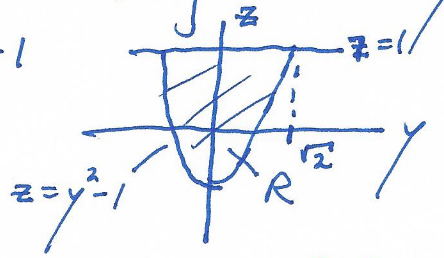
5) (10 points) Let S be the solid region which is bounded on the sides and top by the planes where $x = 0$ and $x + z = 1$ and on the bottom by the parabolic cylinder where $z = y^2 - 1$. Compute the volume of S.

Bade Sadaka



Some sketch - better than this, I hope. 3 pts.
If no sketch, absorb points into limit.

Project into yz-plane:



2 pts.

$$\text{Volume} = \iiint_S dV =$$

$$\iint_R \left[\int_0^{1-z} dx\right] dA =$$

$$\int_{-\sqrt2}^{\sqrt2} \int_{y^2-1}^{1} \int_0^{1-z} dx\,dz\,dy = \quad 3\text{ pts.}$$

$$\int_{-\sqrt2}^{\sqrt2} \int_{y^2-1}^{1} (1-z)\,dz\,dy = \int_{-\sqrt2}^{\sqrt2} \left(z - \frac{z^2}{2}\right)\Big|_{z=y^2-1}^{z=1} dy$$

$$= \int_{-\sqrt2}^{\sqrt2} \frac12 - (y^2-1) + \frac{(y^2-1)^2}{2}\,dy =$$

(see next page)

$$2\int_0^{\sqrt{2}} \frac{1}{2} - y^2 + 1 + \frac{y^4 - 2y^2 + 1}{2}\, dy =$$

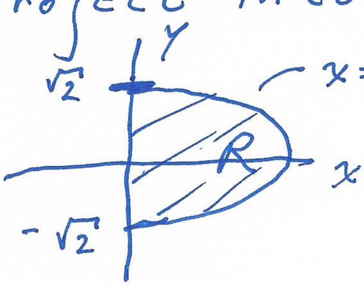
$$2\int_0^{\sqrt{2}} 2 - 2y^2 + \frac{1}{2}y^4\, dy =$$

$$2\left(2y - \frac{2y^3}{3} + \frac{y^5}{10}\right)\Big|_0^{\sqrt{2}} =$$

$$2\left(2\sqrt{2} - \frac{2\cdot 2^{3/2}}{3} + \frac{2^{5/2}}{10}\right) \approx 3.017.$$

$$= \frac{32\sqrt{2}}{15} \leftarrow \text{2pts.}$$

Project into xy-plane: $x + z = 1$. $z = y^2 - 1$. $x + y^2 - 1 = 1$. $x = 2 - y^2$.

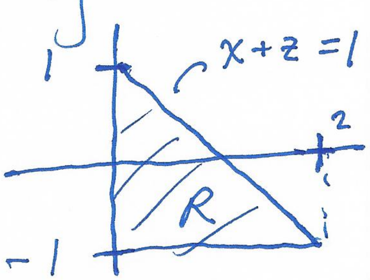


$$\iiint_S dV = \iint_R \left[\int_{y^2-1}^{1-x} dz\right] dA$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{2-y^2} \int_{y^2-1}^{1-x} dz\, dx\, dy = \ldots$$

$$= 2\int_0^{\sqrt{2}} 2 - 2y^2 + \frac{1}{2}y^4\, dy \approx 3.017.$$

Project into xz-plane: $\iiint_S dV =$



$$\iint_R \left[\int_{-\sqrt{1+z}}^{\sqrt{1+z}} dy\right] dA =$$

$$\int_0^2 \int_{-1}^{1-x} \int_{-\sqrt{1+z}}^{\sqrt{1+z}} dy\, dz\, dx = \ldots = \frac{4}{3}\int_0^2 (2-x)^{3/2}\, dx$$

$$= \frac{32}{15}\sqrt{2}.$$

Martsinkovsky

$0 \le \theta \le \frac{\pi}{2}$. $0 \le \phi \le \frac{\pi}{2}$.

6) (8 points) Let S be the solid region in the 1st octant (i.e., where $x \ge 0$, $y \ge 0$, and $z \ge 0$) in $\mathbb{R}^3$ which is contained within the sphere where $x^2 + y^2 + z^2 = 16$, bounded by the cones where $z = \sqrt{x^2+y^2}$ and $z\sqrt{3} = \sqrt{x^2+y^2}$, and bounded by the planes with equations $y = x$ and $y\sqrt{3} = x$. Find the volume of S.

$\rho = 4$. 1 pt.

$z = r = \rho \sin\phi$. $\quad \sqrt{3}\,\rho\cos\phi = \rho\sin\phi$.

$\rho\cos\phi$ $\quad \tan\phi = 1$. $\quad \tan\phi = \sqrt{3}$.

$\phi = \pi/4$. $\quad \phi = \pi/3$. 2 pts.

$y = x$.

$r\sin\theta = r\cos\theta$.

$\tan\theta = 1$. $\theta = \pi/4$ 2 pts.

$y\sqrt{3} = x$.

$\sqrt{3}\, r\sin\theta = r\cos\theta$.

$\tan\theta = \frac{1}{\sqrt{3}}$. $\theta = \pi/6$.

2 pts.

$$\text{Volume} = \iiint_S dV = \int_{\pi/6}^{\pi/4}\int_{\pi/4}^{\pi/3}\int_0^4 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_{\pi/6}^{\pi/4}\int_{\pi/4}^{\pi/3} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=4} d\phi\, d\theta = \frac{64}{3}\int_{\pi/6}^{\pi/4} -\cos\phi\Big|_{\pi/4}^{\pi/3} d\theta$$

1 pt. $\quad \approx 1.1567$

$$= \frac{64}{3}\left(-\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)\right)\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{64}{3}\left(\frac{\sqrt{2}-1}{2}\right)\frac{\pi}{12}$$

Beasley

7) (8 points) Find the mass of the solid right circular cylinder where $-2 \le z \le 2$ and $x^2 + y^2 \le 4$, if the density of the solid region is given by $\delta(x, y) = x^2 + y^2$ kg/m^3. Here x, y, and z are in meters.

$dm = \delta\, dV = r^2 \cdot r\,dr\,d\theta\,dz = r^3\,dr\,d\theta\,dz$.

$$\text{Mass} = \iiint_S dV = \int_{-2}^{2}\int_0^{2\pi}\int_0^2 \underbrace{r^3\,dr\,d\theta\,dz}_{4\text{ pts.}}$$

1 pt. 1 pt. 1 pt.

$$= \int_{-2}^{2}\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta\,dz = \int_{-2}^{2}\int_0^{2\pi} 4\,d\theta\,dz$$

$$= 4(2\pi)4 = 32\pi \text{ kg}.$$

1 pt.

Wish we had written $\delta(x, y, z)$ here.

Massey

8) (8 points) Let $f(x, y, z) = x^2 + y^3 + z^4$ and $\mathbf{F} = \vec{\nabla} f$. Find the line integral of $\mathbf{F}$ along the oriented curve, consisting of four line segments, which go from $(1, 0, 0)$ to $(1, 2, 5)$, then from $(1, 2, 5)$ to $(2, -3, 7)$, then from $(2, -3, 7)$ to $(-4, 6, -7)$, and then from $(-4, 6, -7)$ to $(0, 0, 1)$.

C

Fund. Thm. of Line Integrals:

$$\int_C (\vec{\nabla} f)\cdot d\underline{r} = f(0,0,1) - f(1,0,0) \leftarrow \text{7pts.}$$

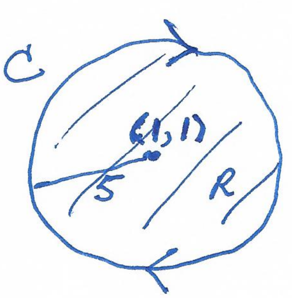
$$= 1 - 1 = 0. \leftarrow \text{1pt.}$$

If done ~~correctl~~ by definition, 2 pts. per piece.

Massey

9) (8 points) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (\underbrace{x^2 - y}_{P},\ \underbrace{y^2 + x}_{Q})$ and C is the circle of radius 5 centered at $(1, 1)$ and oriented clockwise.

Green's Thm.: $\displaystyle\int_{-C} \underline{F}\cdot d\underline{r} = \iint_R (Q_x - P_y)\, dA.$



So, $\displaystyle\int_C \underline{F}\cdot d\underline{r} = \iint_R (P_y - Q_x)\, dA$

$$= \iint_R (-1-1)\, dA = -2\left(\begin{matrix}\text{area}\\ \text{of } R\end{matrix}\right)$$

$$= -2\pi(5)^2 = -50\pi.$$

Fernando

10) Consider the parameterization $\mathbf{r}(u,v) = (u^2+v, u+v, uv)$, where $1 \le u \le 2$ and $0 \le v \le 1$, and let M be the surface parameterized by $\mathbf{r}$.

a) (3 points) We want to orient M by using $\dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ for the positive direction. Show that this is possible by showing that $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ (the zero vector) provided that $1 \le u \le 2$ and $0 \le v \le 1$.

$$\underline{r}_u = (2u, 1, v). \quad \underline{r}_v = (1, 1, u).$$

$$\underbrace{\underline{r}_u \times \underline{r}_v = \begin{vmatrix} i & j & k \\ 2u & 1 & v \\ 1 & 1 & u \end{vmatrix} = (u-v, -(2u^2-v), 2u-1).}_{2\text{ pts.}}$$

$$\underline{r}_u \times \underline{r}_v \neq \underline{0} \quad \text{since} \quad 2u-1 \neq 0, \text{ because } \underbrace{u \neq \tfrac{1}{2}}_{1 \le u \le 2.}.$$

1 pt.

b) (5 points) Orient M as in part (a). Consider the vector field $\mathbf{F}(x,y,z) = (y-x,\ z,\ 0)$. Set up, but **do not evaluate** an iterated integral, in terms of u and v for the flux integral $\int\int_M \mathbf{F}\cdot\mathbf{n}\,dS$ of $\mathbf{F}$ through M. You should "simplify" your iterated integral by evaluating any dot products or cross products, until all that remains is an iterated integral of a polynomial in the variables u and v.

$$\iint_M \underline{F} \cdot \underline{n}\, dS = \underbrace{\int_1^2 \int_0^1 \underline{F}(\underline{r}(u,v)) \cdot (\underline{r}_u \times \underline{r}_v)\, dv\, du}_{2\text{ pts.}}$$

$$= \underbrace{\int_0^2 \int_0^1 (u+v-u^2-v, uv, 0) \cdot (u-v, -2u^2+v, 2u-1)\, dv\, du}_{2\text{ pts.}}$$

$$= \underbrace{\int_0^2 \int_0^1 (u-u^2)(u-v) + uv(-2u^2+v)\, dv\, du.}_{1\text{ pt.}}$$

Lakshmibai

should be bold.

11) (8 points) Let $\mathbf{V}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ be a continuously differentiable velocity vector field, in m/s, where x, y, and z are measured in meters.

Suppose it is known that $Q(x, y, z) = y^2 + e^z$ and $R(x, y, z) = z^3 + \sin x$, in meters per second. Furthermore, suppose that the flux of V is measured through a large number of closed surfaces, with the result that the flux is always $0\ \mathrm{m}^3/\mathrm{s}$. From this, assume that the flux is always $0\ \mathrm{m}^3/\mathrm{s}$ through every reasonably nice closed surface. Give one possible function P that would make this true.

Divergence Thm.

5 pts.

$$\underbrace{\iint_{\partial E} \underline{V} \cdot \underline{n}\, dS}_{\text{Want this to be 0 always.}} = \iiint_E (\vec{\nabla} \cdot \underline{V})\, dV.$$

different

Require $\vec{\nabla} \cdot \underline{V} = 0$.

$$\vec{\nabla} \cdot \underline{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \frac{\partial P}{\partial x} + 2y + 3z^2 = 0. \leftarrow 1 \text{ pt.}$$

$$\frac{\partial P}{\partial x} = -2y - 3z^2.$$

$$P = \int (-2y - 3z^2)\, dx = -2xy - 3xz^2 + A(y, z).$$

So, $\underbrace{P = -2xy - 3xz^2}_{2\text{ pts.}}$ is one such P.

Mickler

12) Suppose that $\mathbf{F}(x, y, z) = (ze^x,\ 2x + y^3 + 7z,\ e^x + 3y + \sin z)$ is a force field on $\mathbb{R}^3$, measured in Newtons, where x, y, and z are measured in meters.

a) (2 points) Calculate the curl, $\vec{\nabla} \times \mathbf{F}$, of $\mathbf{F}$.

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^x & 2x+y^3+7z & e^x+3y+\sin z \end{vmatrix} =$$

2 pts.

$$(3-7,\ -(e^x - e^x),\ 2-0) = (-4, 0, 2)\ N/m.$$

We continue to use the vector field $\mathbf{F}$ from above. Suppose that M is a surface, with boundary ∂M, in $\mathbb{R}^3$ about which you have the following data: M is contained in the plane P where $x + 2y + 3z = 6$, the area of M is 7 square meters, and the positive direction on M is chosen to point upwards, away from the origin (i.e., is chosen to have a positive z component).

b) (1 point) To give the boundary ∂M the orientation that is compatible with the orientation on M, should you orient ∂M clockwise or counterclockwise, if you are looking downwards from above the plane P, towards the origin?

Counterclockwise, 1 pt.

c) (4 points) Giving ∂M its compatible orientation, calculate $\int_{\partial M} \mathbf{F} \cdot d\mathbf{r}$.

(You have enough data to answer this. Hint: What is a normal vector to the plane?) $(1, 2, 3)$

Stokes' Thm.

$$\underbrace{\int_{\partial M} \underline{F} \cdot d\underline{r} = \iint_M}_{1\text{ pt}} (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS =$$

1 pt.

$$\iint_M (-4, 0, 2) \cdot \frac{(1, 2, 3)}{\sqrt{1+4+9}}\, dS = \frac{2}{\sqrt{14}} \iint_M dS$$

$$= \frac{2}{\sqrt{14}}(7) = \frac{14}{\sqrt{14}} = \sqrt{14}\ N \cdot m.$$

2 pts.

d) (1 point) Physically, what does $\int_{\partial M} \mathbf{F} \cdot d\mathbf{r}$ give you?

The work done by $\underline{F}$ on an object moving along ∂M. 1 pt.