

0.1 The Spectral Theorem for Hermitian Operators

- An important class of linear operators is the class of operators that are their own adjoint:
- **Definition:** If $T : V \rightarrow V$ is a linear transformation and T^* exists, we say T is Hermitian (or self-adjoint) if $T = T^*$, and that T is skew-Hermitian if $T = -T^*$.
 - We extend this definition to matrices in the natural way: we say a matrix A is (skew)-Hermitian if $A = [T]_{\beta}^{\beta}$ for some basis β of V and some (skew)-Hermitian linear transformation T .
 - As we showed above, the matrix associated to T^* is A^* , the conjugate-transpose of A , so A is Hermitian precisely when $A = A^*$ and A is skew-Hermitian precisely when $A = -A^*$.
 - If A is a matrix with real entries, then A is Hermitian if and only if $A = A^T$ (i.e., A is a symmetric matrix), and A is skew-Hermitian if and only if $A = -A^T$ (i.e., A is a skew-symmetric matrix).
- Hermitian linear operators (and Hermitian matrices) have a variety of very nice properties. Among the most fundamental of these properties is that all of their eigenvalues are real, and that they are diagonalizable:
- **Theorem** (Properties of Hermitian Operators): Suppose V is a finite-dimensional inner product space and $T : V \rightarrow V$ is a Hermitian linear transformation. Then the following hold:
 1. For any $\mathbf{v} \in V$, $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number.
 - **Proof:** We have $\langle T(\mathbf{v}), \mathbf{v} \rangle = \langle \mathbf{v}, T^*(\mathbf{v}) \rangle = \langle \mathbf{v}, T(\mathbf{v}) \rangle = \overline{\langle T(\mathbf{v}), \mathbf{v} \rangle}$, so $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is equal to its complex conjugate, hence is real.
 2. All eigenvalues of T are real numbers.
 - **Proof:** Suppose λ is an eigenvalue of T with eigenvector $\mathbf{v} \neq \mathbf{0}$.
 - Then $\langle T(\mathbf{v}), \mathbf{v} \rangle = \langle \lambda \mathbf{v}, \mathbf{v} \rangle = \lambda \langle \mathbf{v}, \mathbf{v} \rangle$ is real. Since \mathbf{v} is not the zero vector we conclude that $\langle \mathbf{v}, \mathbf{v} \rangle$ is a nonzero real number, so λ is also real.
 3. Eigenvectors of T with different eigenvalues are orthogonal.
 - **Proof:** Suppose that $T\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ and $T\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$.
 - Then $\lambda_1 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle T\mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, T^*\mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \lambda_2 \mathbf{v}_2 \rangle = \lambda_2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ since λ_2 is real. But since $\lambda_1 \neq \lambda_2$, this means $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$.
 4. Every generalized eigenvector of T is an eigenvector of T .
 - **Proof:** We show by induction that if $(T - \lambda I)^k \mathbf{w} = \mathbf{0}$ then in fact $(T - \lambda I) \mathbf{w} = \mathbf{0}$.
 - For the base case we take $k = 2$, so that $(\lambda I - T)^2 \mathbf{w} = \mathbf{0}$. Then since λ is an eigenvalue of T and therefore real, we have

$$\begin{aligned} \mathbf{0} = \langle (T - \lambda I)^2 \mathbf{w}, \mathbf{w} \rangle &= \langle (T - \lambda I) \mathbf{w}, (T - \lambda I)^* \mathbf{w} \rangle \\ &= \langle (T - \lambda I) \mathbf{w}, (T^* - \bar{\lambda} I) \mathbf{w} \rangle \\ &= \langle (T - \lambda I) \mathbf{w}, (T - \lambda I) \mathbf{w} \rangle \end{aligned}$$
 and thus the inner product of $(T - \lambda I) \mathbf{w}$ with itself is zero, so $(T - \lambda I) \mathbf{w}$ must be zero.
 - For the inductive step, observe that $(T - \lambda I)^{k+1} \mathbf{w} = \mathbf{0}$ implies $(T - \lambda I)^k [(T - \lambda I) \mathbf{w}] = \mathbf{0}$, and therefore by the inductive hypothesis this means $(T - \lambda I) [(T - \lambda I) \mathbf{w}] = \mathbf{0}$, or equivalently, $(T - \lambda I)^2 \mathbf{w} = \mathbf{0}$. Applying the result for $k = 2$ from above yields $(T - \lambda I) \mathbf{w} = \mathbf{0}$, as required.
- Using these results we can establish a fundamental result called the spectral theorem:
- **Theorem** (Spectral Theorem): Suppose V is a finite-dimensional inner product space over \mathbb{R} or \mathbb{C} and $T : V \rightarrow V$ is a Hermitian linear transformation. Then V has an orthonormal basis β of eigenvectors of T , so in particular, T is diagonalizable.
 - The equivalent formulation for Hermitian matrices is: every Hermitian matrix A can be written as $A = U^{-1}DU$ where D is a real diagonal matrix and U is a unitary matrix (i.e., satisfying $U^* = U^{-1}$).
 - **Proof:** By the theorem above, every eigenvalue of T is real hence lies in the scalar field.

- Then every generalized eigenvector of T is an eigenvector of T , and so since V has a basis of generalized eigenvectors, it has a basis of eigenvectors and is therefore diagonalizable.
 - To finish the proof, start with a basis for each eigenspace, and then apply Gram-Schmidt, yielding an orthonormal basis for each eigenspace.
 - Since T is diagonalizable, the union of these bases is a basis for V : furthermore, each of the vectors has norm 1, and they are all orthogonal by the orthogonal result above.
 - By construction, each vector is orthogonal to the others in its eigenspace, and by the observation above it is also orthogonal to the vectors in the other eigenspaces, so we obtain an orthonormal basis β of eigenvectors of T .
 - **Remark:** In fact, the converse of this theorem is also true: if V has an orthonormal basis of eigenvectors of T , then T is Hermitian.
- **Remark:** The set of eigenvalues of T is called the spectrum of T . The spectral theorem shows that V is the direct sum of the eigenspaces of T , meaning that the action of T on V can be decomposed into simple pieces (acting as scalar multiplication), with one piece coming from each piece of the spectrum. (This is the reason for the name of the theorem.)
 - As a corollary we obtain the following extremely useful computational fact:
 - **Corollary:** Every real symmetric matrix has real eigenvalues and is diagonalizable over the real numbers.
 - **Proof:** This follows immediately from the spectral theorem since a real symmetric matrix is Hermitian.
 - **Example:** The real symmetric matrix $A = \begin{bmatrix} 3 & 6 \\ 6 & 8 \end{bmatrix}$ has eigenvalues $\lambda = -1, 12$ and has $A = UDU^{-1}$ where $D = \begin{bmatrix} -1 & 0 \\ 0 & 12 \end{bmatrix}$ and $U = \frac{1}{\sqrt{13}} \begin{bmatrix} -3 & 2 \\ 2 & 3 \end{bmatrix}$.
 - **Example:** The Hermitian matrix $A = \begin{bmatrix} 6 & 2-i \\ 2+i & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1, 7$ and has $A = UDU^{-1}$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$ and $U = \frac{1}{\sqrt{30}} \begin{bmatrix} 5 & 2-i \\ 2+i & -5 \end{bmatrix}$.
 - We will remark that although real symmetric matrices are diagonalizable (and complex Hermitian matrices are diagonalizable), it is *not* true that complex symmetric matrices are always diagonalizable.
 - For example, the complex symmetric matrix $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ is not diagonalizable. This follows from the observation that its trace and determinant are both zero, but since it is not the zero matrix, the only possibility for its Jordan form is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
 - We also remark that most of these results also extend to the class of skew-Hermitian operators (having the property that $T^* = -T$), with appropriate minor modifications.
 - For example, every eigenvalue of a skew-Hermitian operator is a pure imaginary number (i.e., of the form ai for some real number a), and every skew-Hermitian operator is diagonalizable over \mathbb{C} via an orthonormal basis of eigenvectors.
 - All of these statements follow immediately from the simple observation that T is skew-Hermitian if and only if iT is Hermitian.