E. Dummit's Math 5111 ~ Algebra I, Fall 2019 ~ Homework 12, due Dec 4th.

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. The word *show* is synonymous with *prove*.

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1. (Calculations, I) Calculate (no justification required):

   (a) Find the Jordan canonical forms of $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

   (b) Find the Jordan canonical form of the $n \times n$ "all 1s" matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ over $\mathbb{Q}$ and over $\mathbb{F}_p$ (the field with p elements, p prime). [For the latter, the answer will depend on whether p divides n or not.]

   (c) Find the minimal and characteristic polynomials of the $n \times n$ Jordan block with eigenvalue λ.

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2. (Calculations, II) Suppose the characteristic polynomial of the 5×5 matrix A is $p(t) = t^3(t-1)^2$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   (b) List all possible Jordan canonical forms of A up to equivalence.

   (c) If $\ker(A)$ and $\ker(A - I)$ are both 2-dimensional, what is the Jordan canonical form of A?

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3. (Minimal Polynomial, II) Suppose A is an $n \times n$ matrix over $\mathbb{C}$ and that λ is an eigenvalue of A.

   (a) Prove that λ is a root of the minimal polynomial of A. Deduce that the minimal polynomial and the characteristic polynomial have the same roots.

   (b) Show that the exponent of $t - \lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ in the Jordan canonical form of T.

   (c) Show that the minimal polynomial of a 2×2 matrix uniquely determines its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 2×2 matrices with minimal polynomials $m(t) = t^2 - t$, $t^2 + 1$, and $t - \lambda$.

   (d) Show the minimal and characteristic polynomials of a 3×3 matrix together uniquely determine its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 3×3 matrices with $(m(t), p(t))$ equal to (t, t^3), (t^2, t^3), (t^3, t^3), and $(t^2 - 1, t^3 - t^2 - t + 1)$.

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4. Let $A \in M_{n \times n}(\mathbb{C})$. Show that the following are equivalent:

   (a) The ranks of A and A^2 are equal.

   (b) The multiplicity of 0 as a root of the minimal polynomial of A is at most 1.

   (c) There is an $n \times n$ matrix X such that $AXA = A$, $XAX = X$, and $AX = XA$.

   (d) If there are any, every Jordan block with eigenvalue 0 in the Jordan canonical form of A has size 1.

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5. (Matrix Orders, I) Let F be an algebraically closed field and $G = GL_n(F)$ be the group of $n \times n$ invertible matrices over F.

   (a) If A has finite order k in G, prove that all eigenvalues of A are kth roots of unity in F (i.e., elements $\alpha \in F$ for which $\alpha^k = 1$). [Hint: Use 3(a) and the fact that the minimal polynomial divides any polynomial annihilating A.]

   (b) Show that if $\text{char}(F) = 0$, then the only Jordan blocks of finite order are the 1×1 blocks.

   (c) Deduce that if $\text{char}(F) = 0$, then A has finite order if and only if it is diagonalizable and all its eigenvalues are roots of unity.

   (d) [Optional] If $\text{char}(F) > 0$, prove that A has finite order if and only if all its eigenvalues are roots of unity.

   - Remark: If F is not algebraically closed, the results of this problem still essentially hold, provided one works with the diagonalization and eigenvalues over a field extension K/F containing all the eigenvalues of G (e.g., the algebraic closure of F), since the order of $G \in GL_n(F)$ is the same in $GL_n(K)$.

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6. (Matrix Orders, II) Let F be a field and $G = GL_n(F)$.

   (a) If $\text{char}(F) \neq 2$, show that G has precisely n conjugacy classes of elements of order 2. [Hint: If g has order 2, then $g^2 - I = 0$.]

   (b) If $\text{char}(F) = 2$, show that G has precisely $\lfloor n/2 \rfloor$ (the greatest integer $\leq n/2$) conjugacy classes of elements of order 2.

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7. (Matrix Orders, III) Let K be a field where -1 is not a square, and let $G = GL_2(K)$.

   (a) If $g \in G$, show that g has order 4 if and only if $\det(g) = 1$ and $\text{tr}(g) = 0$. [Hint: Consider the eigenvalues of g.]

   (b) Find an explicit element $g \in G$ of order 4.

   (c) Suppose there exist elements $a, b \in K$ with $a^2 + b^2 = -1$. Show that G contains two elements g, h of order 4 such that gh also has order 4.
