E. Dummit's Math 5111 ~ Algebra I, Fall 2019 ~ Homework 11, due Nov 27th.

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. The word *show* is synonymous with *prove*. This assignment has six problems and two pages.

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1. Calculate (you do not need to include justification):
   (a) The eigenvalues and eigenvectors of $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.
   (b) Diagonalizations of the diagonalizable matrices in part (a).
   (c) The eigenvalues and eigenvectors of the $n \times n$ "all 1s" matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ over $\mathbb{Q}$.
   (d) The eigenvalues and eigenvectors of the left and right shift operators on the space of infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

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2. (Hermitian Operators) Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is a linear transformation possessing an adjoint T^*. We say T is <u>Hermitian</u> (or <u>self-adjoint</u>) if $T = T^*$, and that T is <u>skew-Hermitian</u> if $T = -T^*$.
   (a) Show that T is Hermitian if and only if iT is skew-Hermitian.
   (b) Show that $T + T^*$, T^*T, and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.
   (c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2.
   (d) Suppose T is skew-Hermitian. Prove that $\langle T\mathbf{v}, \mathbf{v}\rangle$ is a purely imaginary number for any vector $\mathbf{v}$, and conclude that all eigenvalues of a skew-Hermitian transformation are purely imaginary.

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3. (Isometries) Suppose V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$ (not necessarily finite-dimensional) and $T : V \to V$ is linear. We say T is a "distance-preserving" map on V if $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v}$ in V, and we say T is an "angle-preserving" map on V if $\langle \mathbf{v}, \mathbf{w}\rangle = \langle T\mathbf{v}, T\mathbf{w}\rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.
   (a) Prove the "polarization identities": if $F = \mathbb{R}$ prove that $\langle \mathbf{v}, \mathbf{w}\rangle = \frac{1}{4}||\mathbf{v}+\mathbf{w}||^2 - \frac{1}{4}||\mathbf{v}-\mathbf{w}||^2$ while if $F = \mathbb{C}$ prove that $\langle \mathbf{v}, \mathbf{w}\rangle = \frac{1}{4}\sum_{k=1}^4 i^k \left|\left|\mathbf{v} + i^k\mathbf{w}\right|\right|^2$.
   (b) Prove that T is distance-preserving if and only if it is angle-preserving. [Hint: Use (a).]

   A map $T : V \to V$ satisfying the distance- and angle-preserving conditions is called a (linear) <u>isometry</u>.
   (c) Show that isometries are one-to-one and that they preserve orthonormal sets.
   (d) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.
   (e) Deduce that the isometries of $\mathbb{C}^n$ (with its usual inner product) are given by left-multiplication by a unitary matrix (i.e., one with $A^*A = I$).

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4. (Commuting Operators) Suppose V is a nonzero finite-dimensional $\mathbb{C}$-vector space.
   (a) If S and T are commuting linear operators on V, prove that S maps each eigenspace of T into itself.
   (b) Now suppose $T_1, \dots T_k$ are pairwise commuting linear operators on V. Show that $T_1, \dots T_n$ have a common eigenvector in V.
   (c) Prove that there exists a filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ where $\dim(V_i) = i$ and V_i is mapped into itself by each of $T_1, \dots T_k$.

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5. (Minimal Polynomials) Suppose V is finite-dimensional and $T : V \to V$. We say the polynomial $q(x) \in F[x]$ annihilates T if $q(T) = 0$.

   (a) Show that the set of polynomials in $F[x]$ annihilating T is a vector space.

   We define the minimal polynomial of T to be the monic polynomial $m(x) \in F[x]$ of smallest positive degree annihilating T. For example, the minimal polynomial of the identity transformation is $m(x) = x - 1$.

   (b) Show that every polynomial that annihilates T is divisible by the minimal polynomial (hint: use polynomial division), and conclude that the minimal polynomial divides the characteristic polynomial.

   (c) Part (b) gives a (moderately effective) way to find the minimal polynomial, namely, test all possible monic divisors of the characteristic polynomial. Using this method or otherwise, find the minimal polynomials of the matrices $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.

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6. (Cyclic and Irreducible Subspaces) Suppose $\dim_F(V) = n$ and $T : V \to V$. We say that V is T-cyclic if there exists a vector $\mathbf{v} \in V$ such that $V = \text{span}(\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, T^3\mathbf{v}, \dots)$, and we say that V is T-irreducible if it has no nontrivial proper T-stable subspaces (i.e., if $T(W) \subseteq W$ implies $W = 0$ or $W = V$).

   (a) If V is T-irreducible, show that V is T-cyclic. [Hint: Choose $\mathbf{v} \neq \mathbf{0}$ and consider $W = \text{span}(\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, T^3\mathbf{v}, \dots)$.]

   (b) If V is T-cyclic and $\mathbf{w} \in V$, show that there exists a polynomial q such that $\mathbf{w} = q(T)\mathbf{v}$.

   (c) If V is T-cyclic, show that $\beta = \{\mathbf{v}, T\mathbf{v}, \dots, T^{n-1}\mathbf{v}\}$ is a basis of V and $[T]_\beta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$ for some elements $a_0, a_1, \dots, a_{n-1} \in F$.

   (d) With notation as in part (c), show that the characteristic polynomial of T is $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

   (e) With notation as in part (c), show that the minimal polynomial of T is also equal to $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. [Hint: If $f(T) = 0$ then $f(T)\mathbf{v} = 0$. Use the fact that β is a basis to see that such an f has degree at least n, and then use 5(b).]

   (f) [Optional] Conversely, suppose that the minimal and characteristic polynomials of T are equal. Prove that V is T-cyclic. [Hint: Show that there must exist a vector $\mathbf{v}$ such that no polynomial f of degree $\leq n - 1$ has $f(T)\mathbf{v} = 0$.]

   - Remark: With some additional effort, it can be proven that any finite-dimensional vector space decomposes as a direct sum of T-irreducible spaces. (One approach is to prove that any T-stable subspace W has a T-stable complement W' such that $W \oplus W' = V$, which inside $\mathbb{C}$ can be done by using adjoint transformations to reduce the result to the existence of orthogonal complements.) This fact along with the results above establish the existence of the rational canonical form of T, which says that there exists a basis β of V for which the corresponding matrix $[T]_\beta$ is a direct sum of "companion matrices" of the form given in part (c).
