
Part I: Calculation Problems

1. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) irreflexive, (vi) an equivalence relation, (vii) a partial ordering, and (viii) a total ordering.
   - (a) $R = \{(1,1),\ (2,1),\ (2,2)\}$ on the set $\{1,2\}$.
   - (b) $R = \{(1,2),\ (2,1)\}$ on the set $\{1,2\}$.
   - (c) $R = \{(1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,3),\ (3,4),\ (4,3),\ (4,4)\}$ on the set $\{1,2,3,4\}$.
   - (d) The divisibility relation on the set $\{2,-3,4,-5,6\}$.
   - (e) The divisibility relation on the set $\{2,-4,-12,36\}$.
   - (f) The relation R on $\mathbb{Z}$ with $a\ R\ b$ precisely when $|a| \equiv |b|$ modulo 5.
   - (g) The relation R on $\mathbb{R}$ with $a\ R\ b$ precisely when $ab > 0$.

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2. For each of the following functions $f : A \to B$, determine whether (i) f is one-to-one, (ii) f is onto, (iii) f is a bijection.
   - (a) $f(x) = 2x$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
   - (b) $f(n) = 2n$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.
   - (c) $f(x) = \dfrac{x}{x-1}$ from $A = \mathbb{R}\backslash\{1\}$ to $B = \mathbb{R}$.
   - (d) $f(x) = x^3$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
   - (e) $f = \{(1,2),(2,3),(3,4),(4,1)\}$ from $\{1,2,3,4\}$ to itself.
   - (f) $f = \{(1,3),(2,4),(3,1),(4,4)\}$ from $\{1,2,3,4\}$ to itself.

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3. Identify the ordered pairs in the equivalence relation that corresponds to the partition $\{1,2,4\},\{3,5\},\{6\}$ of $\{1,2,3,4,5,6\}$.

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4. Show $R = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : |x| = |y|\}$ is an equivalence relation and list the equivalence classes of 0, 1, 2, -2, and 4.

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5. Find the coefficients of x^{18} in $(2x+1)^{24}$, in $(x^2+x)^{13}$, and in $(x^3 - 2/x)^{14}$.

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6. A U.S. social security number (SSN) is a 9-digit string of the form abc-de-fghi. Find:
   - (a) The number of SSNs where all of the digits are odd.
   - (b) The number of SSNs that have no 0s.
   - (c) The number of SSNs that have at least one 0.
   - (d) The number of SSNs that have exactly four 0s.
   - (e) The number of SSNs that have at least one 0 or 8.
   - (f) The number of SSNs with no repeated digits.
   - (g) The number of SSNs with no double digits (i.e., no 00, 11, etc.).
   - (h) The number of SSNs where abc + de = fghi.
   - (i) The number of SSNs where abc, de, and fghi are all multiples of 3.
   - (j) The number of SSNs where abc · de · fghi is a multiple of 3.
   - (k) The number of SSNs where each digit is $\geq$ the one to its left (e.g., 997-64-3100).

Part II: Proof Problems

1. Suppose $f : A \to B$ is a function.

   (a) If f is one-to-one, show that there is a bijection between A and $\mathrm{im}(f)$. Deduce that $\#A = \#\mathrm{im}(f)$.

   (b) If A and B are both finite and $\#A = \#B$, show that f is one-to-one implies that f is onto.

   (c) Show that (b) is false for infinite sets by giving a function $f : \mathbb{R} \to \mathbb{R}$ where f is one-to-one but not onto.

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2. A real number is algebraic if it is a root of a nonzero polynomial $p(x)$ with integer coefficients, and it is transcendental if it is not the root of any such polynomial.

   (a) Let S_n be the set of all roots of nonzero polynomials of degree at most n whose coefficients are integers and at most n in absolute value. Show that S_n is finite and that the set of algebraic numbers is $\cup_{n\geq 1} S_n$.

   (b) Show that the set of algebraic numbers is countable. Deduce there are uncountably many transcendental numbers.

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3. Let A, B, C be sets, R be a relation, f, g be functions, and m, k, n be nonnegative integers. Prove the following:

   (a) Show that the only equivalence relation R on A that is a function from A to A is the identity relation.

   (b) Prove that if A is countable and B is uncountable, then $B \backslash A$ is uncountable.

   (c) Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 9^k = 10^n$.

   (d) Find the number of partitions of $\{1, 2, \ldots, 2n\}$ into n unordered sets of 2 elements. Deduce that $2^n n!$ divides $(2n)!$.

   (e) Prove that the set $\mathbb{Q} \times \mathbb{Z}$ is countable and that the set $\mathbb{R} \times \mathbb{Z}$ is uncountable.

   (f) Prove that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$, the set of rational numbers strictly between 0 and 1.

   (g) A committee of 3 people is chosen from n mathematicians and n computer scientists. Prove that the number of possible committees is $2\binom{n}{3} + 2n\binom{n}{2}$ and deduce that $2\binom{n}{3} + 2n\binom{n}{2} = \binom{2n}{3}$.

   (h) Prove that if $f : A \to B$ is one-to-one and $S \subseteq A$, then $f^{-1}(f(S)) = S$.

   (i) Prove that if $f : A \to B$ is onto and $T \subseteq B$, then $f(f^{-1}(T)) = T$.

   (j) Prove that there exists a bijection between $(0, 1)$ and $[0, 1]$. [Hint: Use the Cantor-Schroder-Bernstein theorem.]

   (k) Prove that $\displaystyle\sum_{k=1}^{n} k \cdot k! = (n + 1)! - 1$ for every positive integer n.

   (l) Suppose $f : A \to B$ is a bijection. Show that $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ given by $\tilde{f}(S) = \{f(s) : s \in S\}$ is also a bijection.

   (m) If $R, S : A \to B$ are relations, prove that $R^{-1} \cap S^{-1} = (R \cap S)^{-1}$.

   (n) Prove that $\displaystyle\sum_{k=0}^{n} \frac{2^k}{k!(n-k)!} = \frac{3^n}{n!}$.
