
The answers to these problems are only sketched out, and are not given in full detail. They are intended more as outlines for the complete solutions, which should be straightforward to fill out for someone who has already tried working through the problems. Many problems have more than one possible approach, so if your approach is not the one given here, it may still be correct.

Part I: Calculation Problems

1.

#	Reflexive	Symmetric	Transitive	Antisymmetric	Irreflexive	Equiv Rel	Partial	Total
(a)	Yes	No	Yes	Yes	No	No	Yes	Yes
(b)	No	Yes	No	No	Yes	No	No	No
(c)	Yes	Yes	Yes	No	No	Yes	No	No
(d)	Yes	No	Yes	Yes	No	No	Yes	No
(e)	Yes	No	Yes	Yes	No	No	Yes	Yes
(f)	Yes	Yes	Yes	No	No	Yes	No	No
(g)	No (0)	Yes	Yes	No	No	No	No	No

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2. (a) f is one-to-one, onto, and a bijection since it has an inverse $f^{-1}(x) = x/2$.
(b) f is one-to-one but not onto since $\text{im}(f)$ is only the even integers.
(c) f is one-to-one but not onto since its image misses 1.
(d) f is one-to-one, onto, and a bijection since it has an inverse $f^{-1}(x) = x^{1/3}$.
(e) f is one-to-one, onto, and a bijection since its inverse is also a function.
(f) f is not one-to-one since $f(2) = f(4)$ and f is not onto since $\text{im}(f)$ misses 2.

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3. $R = \{(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(4,1),(4,2),(4,4),(3,3),(3,5),(5,3),(5,5),(6,6)\}$.

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4. R is reflexive since $|x| = |x|$, R is symmetric since $|x| = |y|$ implies $|y| = |x|$, and R is transitive since $|x| = |y|$ and $|y| = |z|$ imply $|x| = |z|$.
Also, $[0] = \{0\}$, $[1] = \{1, -1\}$, $[2] = [-2] = \{2, -2\}$, $[4] = \{4, -4\}$.

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5. Coefficient of x^{18} in $(2x+1)^{24}$ is $\binom{24}{18}2^{18}$, in $(x^2+x)^{13}$ is $\binom{13}{5}$, and in $(x^3 - 2/x)^{14}$ is $\binom{14}{6} \cdot (-2)^6$.

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6. (a) 5^9
(b) 9^9
(c) $10^9 - 9^9$
(d) $\binom{9}{4} \cdot 9^5$
(e) $10^9 - 8^9$
(f) $10!$
(g) $10 \cdot 9^8$
(h) 10^5
(i) $333 \cdot 33 \cdot 3333$
(j) $10^9 - 667 \cdot 67 \cdot 6667$
(k) $\binom{18}{9}$ by stars-and-bars

Part II: Proof Problems

1. (a) The function $f^\dagger : A \to \text{im}(f)$ is one-to-one and onto hence a bijection. Then $\#A = \#\text{im}(f)$ by the definition of cardinality.
(b) If f is one-to-one then by (a), $\#A = \#\text{im}(f)$. Since $\#A = \#B$ and A and B are finite, this means $\text{im}(f) = B$ so f is onto.
(c) Many examples, such as $f(x) = e^x$ or $f(x) = \arctan(x)$.

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2. (a) There are finitely many nonzero polynomials of degree at most n whose coefficients are integers and at most n in absolute value. Each of these polynomials has at most n roots, so the total number of roots is finite. Second statement follows by observing that if α is a root of a poly of degree k whose max coeff is M, then $\alpha \in S_{\max(k,M)}$.
(b) By (a), the set of algebraic numbers is a union of countably many finite sets, so it is countable. Since $\mathbb{R}$ is uncountable, this means that the transcendental numbers are uncountable since otherwise $\mathbb{R}$ would be the union of two countable sets hence countable.

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3. (a) If R is reflexive and a function, then $R(a) = a$ for all $a \in A$, so the only possibility is to have $R(a) = a$ for all $a \in A$. But clearly the identity function is also an equivalence relation, so it is the only one that works.
(b) Note that B is a subset of $A \cup (B \backslash A)$. If A and $B \backslash A$ are countable then their union is also countable, hence any subset is countable. If B is uncountable then this is a contradiction, so B is uncountable.
(c) The result follows immediately from applying the binomial theorem to $(9+1)^n$.
(d) There are $\binom{n}{2}$ ways to choose the first set, $\binom{n-2}{2}$ ways to choose the second set, etc., and $\binom{2}{2}$ ways to choose the last set, total $(2n)!/2^n$. But sets may be permuted arbitrarily, so the number of unordered partitions into pairs is $(2n)!/(2^n \cdot n!)$. The second statement follows since this count is an integer.
(e) As proven in class, the Cartesian product of two countable sets is countable, so $\mathbb{Q} \times \mathbb{Z}$ is countable. Note $\mathbb{R} \times \mathbb{Z}$ contains $\mathbb{R} \times \{1\}$ which is in bijection with $\mathbb{R}$, so it is uncountable.
(f) Both sets are countably infinite. Hence they are both in bijection with the positive integers, and therefore also with each other.
(g) Either all 3 committee members can be chosen from one group, total $2\binom{n}{3}$ ways, or 1 from one and 2 from the other, total $2n\binom{n}{2}$ ways. But the total number is also $\binom{2n}{3}$, so $2\binom{n}{3} + 2n\binom{n}{2} = \binom{2n}{3}$ as claimed.
(h) From homework 8, $S \subseteq f^{-1}(f(S))$. For the reverse, suppose $a \in f^{-1}(f(S))$, so that $f(a) \in f(S)$. Since f is one-to-one, $f(a) = f(b)$ implies $a = b$, so $f(a) \in f(S)$ implies $a \in S$.
(i) From homework 8, $f(f^{-1}(T)) \subseteq T$. For the reverse, suppose $b \in T$. Since f is onto, there exists $a \in A$ with $f(a) = b$, so $a \in f^{-1}(T)$. Hence $b \in f(f^{-1}(T))$.
(j) We find one-to-one maps in each direction. Many choices, but simple ones are $f : (0,1) \to [0,1]$ with $f(x) = x$ and $g : [0,1] \to (0,1)$ with $g(x) = (x+1)/3$. Then by the Cantor-Schröder-Bernstein theorem, there exists a bijection between $(0,1)$ and $[0,1]$.
(k) Induct on n. The base case $n = 1$ is trivial. The inductive step follows from the calculation $\sum_{k=1}^{n+1} k \cdot k! = (n+1) \cdot (n+1)! + \sum_{k=1}^{n} k \cdot k! = (n+1) \cdot (n+1)! + (n+1)! - 1 = (n+2)(n+1)! - 1 = (n+2)! - 1$.
(l) Note f has an inverse g. Then it is straightforward to check that $\tilde{f}$ has an inverse $\tilde{g} : \mathcal{P}(B) \to \mathcal{P}(A)$ with $g(T) = \{g(t) : t \in T\}$.
(m) Observe that $(a,b) \in R^{-1} \cap S^{-1}$ iff $(a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$ iff $(b,a) \in R$ and $(b,a) \in S$ iff $(b,a) \in R \cap S$ iff $(a,b) \in (R \cap S)^{-1}$.
(n) By the binomial theorem, $3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \dfrac{n!}{k!(n-k)!} 2^k$. Dividing by $n!$ yields the result.
