
Part I: Calculation Problems

1. Suppose the logical operator $*$ is defined so that $P * Q = \neg P \wedge Q$. Using a truth table or otherwise, determine whether the following pairs of statements are logically equivalent for arbitrary propositions P, Q, and R:
   (a) $P * (P * P)$ and $\neg P$.
   (b) $(P \Rightarrow Q) \Rightarrow (P * Q)$ and True.
   (c) $P * Q$ and $\neg(Q \Rightarrow P)$.
   (d) $(Q * Q) * P$ and $P \wedge Q$.
   (e) $(P * Q) * R$ and $P * (Q * R)$.
   (f) $(P * Q) \vee R$ and $(R * \neg Q) * (R * P)$.

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2. Suppose that $A = \{1, 2, \{1\}, \{2\}, \{1, 3\}\}$ and $B = \{1, \{1, 2\}, \{1, 3\}, \{2\}\}$. Find the truth value of each statement:
   (a) $1 \in A$.
   (b) $1 \subseteq A$.
   (c) $\{1\} \in B$.
   (d) $\{1\} \subseteq B$.
   (e) $\{1\} \in A \cap B$.
   (f) $\{1\} \subseteq A \cap B$.
   (g) $\{1, 2\} \in A \cup B$.
   (h) $\{1, 2\} \subseteq A \cup B$.
   (i) $\{1, 3\} \in A \cap B$.
   (j) $\{1, 3\} \subseteq A \cap B$.

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3. Suppose that $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 4, 8\}$, and $C = \{2, 3, 5, 7\}$ with universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find:
   (a) $A \cap B$.
   (b) $A \cup C$.
   (c) $A^c \cap C^c$.
   (d) $B^c \cup (A \cap C)$.
   (e) $(B \cap C) \cap (A \cup B^c)$.

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4. In a survey of 100 pet owners, 43 owned a cat, 44 owned a dog, 31 owned a fish, 23 owned a cat and a dog, 3 owned a cat and a fish, and 11 owned a dog and fish. If 1 person owned all three pets, find:
   (a) The number of people who owned a cat or a dog.
   (b) The number of people who owned a cat, but neither a dog nor a fish.
   (c) The number of people who owned a fish and either a cat or dog but not both.
   (d) The number of people who owned none of the three.

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5. Write a negation for each of the following statements:
   (a) $\forall x \forall y \exists z,\ x + y + z > 5$.
   (b) Every integer is a rational number.
   (c) $\forall x \in A\, \forall y \in B,\ x \cdot y \in A \cap B$.
   (d) There is a perfect square that is not even.
   (e) The integer n is a prime number and $n < 10$.
   (f) $\forall \epsilon > 0\, \exists \delta > 0,\ (|x - a| < \delta) \Rightarrow (|x^2 - a^2| < \epsilon)$.
   (g) For any $x \in \mathbb{R}$ there exists an $n \in \mathbb{Z}$ such that $x < n$.
   (h) There exist integers a and b such that $\sqrt[3]{2} = a/b$.

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6. Find the truth values of the following statements, where the universe of discourse is $\mathbb{R}$:
   (a) $\forall x \forall y,\ y \neq x$.
   (b) $\forall x \exists y,\ y \neq x$.
   (c) $\exists x \forall y,\ y \neq x$.
   (d) $\exists x \exists y,\ y \neq x$.
   (e) $\forall x \forall y,\ y^2 \geq x$.
   (f) $\forall x \exists y,\ y^2 \geq x$.
   (g) $\exists x \forall y,\ y^2 \geq x$.
   (h) $\exists x \exists y,\ y^2 \geq x$.

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7. Calculate the following things:
   (a) The gcd and lcm of 256 and 520.
   (b) The gcd and lcm of 921 and 177.
   (c) The gcd and lcm of 2019 and 5678.
   (d) The gcd and lcm of $2^3 3^2 5^4 7$ and $2^4 3^3 5^4 11$.
   (e) The values of $\overline{4} + \overline{6}$, $\overline{4} - \overline{6}$, and $\overline{4} \cdot \overline{6}$ modulo 8.
   (f) The remainder when $12 \cdot 13 \cdot 14 \cdot 15$ is divided by 11.

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8. Decide whether each residue class has a multiplicative inverse modulo m. If so, find it, and if not, explain why not:
   (a) The residue class $\overline{10}$ modulo 25.
   (b) The residue class $\overline{11}$ modulo 25.
   (c) The residue class $\overline{12}$ modulo 25.
   (d) The residue class $\overline{30}$ modulo 42.
   (e) The residue class $\overline{31}$ modulo 42.
   (f) The residue class $\overline{32}$ modulo 42.

Part II: Proof Problems

1. Suppose A, B, and C are arbitrary sets contained in a universal set U. Identify which statements are true and which are false. Then prove the true statements and give a counterexample for the false ones.
   (a) $(A \cup B) \backslash A = B \backslash A$.
   (b) $A \backslash (B \cap C) = (A \backslash B) \cap (A \backslash C)$.
   (c) $(A \cap B)^c \cup B \subseteq (A^c \cup B)^c$.
   (d) $A^c \cap B^c \subseteq (A \backslash B)^c \cap (B \backslash A)^c$.

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2. Write, and then prove, the contrapositive of each of these statements (assume n refers to an integer):
   (a) If a and b are integers, then $3a - 9b \neq 2$.
   (b) Suppose $a, b \in \mathbb{Z}$. If $ab = 1$ then $a \leq 1$ or $b \leq 1$.
   (c) If $5n + 1$ is even, then n is odd.
   (d) If n^3 is odd, then n is odd.
   (e) If n is not a multiple of 3, then n cannot be written as the sum of 3 consecutive integers.
   (f) Suppose p is prime. If p does not divide a and p does not divide b, then p does not divide ab.

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3. Find a counterexample to each of the following statements:
   (a) For any integers a, b, and c, if $a|b$ and $a|c$, then $b|c$.
   (b) If p and q are prime, then $p + q$ is never prime.
   (c) If n is an integer, then $n^2 + n + 11$ is always prime.
   (d) There do not exist integers a and b with $a^2 - b^2 = 23$.
   (e) The sum of two irrational numbers is always irrational.
   (f) If $n > 1$ is an integer, then $\sqrt{n}$ is always irrational.

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4. Prove the following (recall the Fibonacci numbers F_i are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$):
   (a) If F_n is the nth Fibonacci number, prove that $F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$ for every positive integer n.
   (b) Suppose n is an integer. Prove that $2|n$ and $3|n$ if and only if $6|n$.
   (c) Prove that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for every positive integer n.
   (d) Suppose p is a prime and a is a positive integer. If $p|a^2$, prove that $p|a$.
   (e) Prove there do not exist integers a and b such that $a^2 = 33 + 9b$. [Hint: Use the previous part.]
   (f) Prove any two consecutive perfect squares (i.e., the integers k^2 and $(k+1)^2$) are relatively prime.
   (g) Prove that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$ for every positive integer n.
   (h) If $A = \{4a + 6b : a, b \in \mathbb{Z}\}$ and $B = \{2c : c \in \mathbb{Z}\}$, prove that $A = B$.
   (i) Prove that the sum of any four consecutive integers is congruent to 2 modulo 4.
   (j) If p is a prime, prove that $\gcd(n, n+p) > 1$ if and only if $p|n$.
   (k) Suppose $a_1 = 1$ and $a_n = 3a_{n-1} + 4$ for all $n \geq 2$. Prove that $a_n = 3^n - 2$ for every positive integer n.
   (l) If $C = \{6c : c \in \mathbb{Z}\}$ and $D = \{10a + 14b : a, b \in \mathbb{Z}\}$, prove that $C \subseteq D$.
   (m) If n is any positive integer, prove that $n - 1$ is invertible modulo n and its multiplicative inverse is itself.
   (n) Suppose $b_1 = 3$ and $b_n = 2b_{n-1} - n + 1$ for all $n \geq 2$. Prove that $b_n = 2^n + n$ for every positive integer n.
   (o) Using only basic facts about divisibility, prove every integer $n > 1$ has a prime divisor. [Hint: Strong induction.]
   (p) Suppose $c_1 = c_2 = 2$, and for all $n \geq 3$, $c_n = c_{n-1}c_{n-2}$. Prove that $c_n = 2^{F_n}$ for every positive integer n.
   (q) Suppose $d_1 = 2$, $d_2 = 4$, and for all $n \geq 3$, $d_n = d_{n-1} + 2d_{n-2}$. Prove that $d_n = 2^n$ for every positive integer n.
   (r) If a and b are positive integers, prove that $\gcd(a, b) = \text{lcm}(a, b)$ if and only if $a = b$.
