The answers to these problems are only sketched out, and are not given in full detail. They are intended more as outlines for the complete solutions, which should be straightforward to fill out for someone who has already tried working through the problems. Many problems have more than one possible approach, so if your approach is not the one given here, it may still be correct.

Part I: Calculation Problems

1. (a) Not equivalent (b) Not equivalent (c) Equivalent (d) Not equivalent (e) Not equivalent (f) Equivalent

2. (a) True (b) False (c) False (d) True (e) False (f) True (g) False (h) True (i) True (j) False

3. (a) $\{1\}$ (b) $\{1,2,3,5,7,9\}$ (c) $\{4,6,8\}$ (d) $\{3,5,6,7,9\}$ (e) $\emptyset = \{\}$

4. (a) 64 people (b) 18 people (c) 12 people (d) 18 people

5. (a) $\exists x \exists y \forall z,\ x + y + z \leq 5$ (b) There exists an integer that is not a rational number. (c) $\exists x \in A\, \exists y \in B,\ x \cdot y \notin A \cap B$. (d) Every perfect square is even. (e) The integer n is either not prime or $n \geq 10$. (f) $\exists \epsilon > 0\, \forall \delta > 0,\ (|x - a| < \delta) \wedge (|x^2 - a^2| \geq \epsilon)$. (g) There exists an $x \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$, $x \geq n$. (h) For all integers a and b, $\sqrt[3]{2} \neq a/b$.

6. (a) False (b) True (c) False (d) True (e) False (f) True (g) True (h) True

7. (a) By Euclid, gcd 8, lcm $256 \cdot 520/8$. (b) By Euclid, gcd 3, lcm $921 \cdot 177/3$. (c) By Euclid, gcd 1, lcm $2019 \cdot 5678$. (d) gcd $2^3 3^2 5^4$, lcm $2^4 3^3 5^4 7 \cdot 11$. (e) sum is $\overline{2}$, difference is $\overline{6}$, product is $\overline{0}$. (f) Mod 11, product is $1 \cdot 2 \cdot 3 \cdot 4 = 24 \equiv 2$.

8. (a) No, 10 and 25 not relatively prime. (b) Yes, by Euclid, inverse is $\overline{16}$. (c) Yes, by Euclid, inverse is $\overline{23}$. (d) No, 30 and 42 not relatively prime. (e) Yes, by Euclid, inverse is $\overline{19}$. (f) No, 32 and 42 not relatively prime.

Part II: Proof Problems

1. (a) True. Note $x \in (A \cup B) \backslash A$ iff $x \in (A \cup B) \cap A^c$ iff $x \in B \cap A^c$ iff $x \in B \backslash A$.
(b) False. Counterexample: $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$. Then $A \backslash (B \cap C) = \{1,2\}$ while $(A \backslash B) \cap (A \backslash C) = \emptyset$.
(d) False. Counterexample: $A = \{1\}$, $B = \{1,2\}$ with $U = \{1,2\}$. Then $(A \cap B)^c \cup B = \{1,2\}$ while $(A^c \cap B)^c = \{1\}$.
(e) True. Note $(A \backslash B)^c = (A \cap B^c)^c = A^c \cup B$, and similarly $(B \backslash A)^c = A \cup B^c$. If $x \in A^c \cap B^c$ then $x \in A^c \cup B$ and also $x \in A \cup B^c$.

2. (a) If $3a - 9b = 2$, then a and b cannot both be integers. Proof: By contradiction, if a and b are integers, then 3 divides $3a - 9b$ but 3 does not divide 2 (impossible).
(b) If $a > 1$ and $b > 1$, then $ab \neq 1$. Proof: If $a > 1$ and $b > 1$ then $ab > 1$ (e.g., as proven on a homework assignment).
(c) If n is even, then $5n + 1$ is odd. Proof: If $n = 2k$ then $5n + 1 = 10k + 1 = 2(5k) + 1$ is odd by definition.
(d) If n is even then n^3 is even. Proof: If $n = 2k$ then $n^3 = 8k^3 = 2(4k^3)$ is even by definition.
(e) If n is the sum of 3 consecutive integers, then n is a multiple of 3. Proof: If $n = a + (a+1) + (a+2)$ then $n = 3a + 3 = 3(a+1)$ is a multiple of 3.
(f) If p (a prime) divides ab then p divides a or p divides b. This is a fact about prime numbers established in class.

3. There are many examples for each part. Here is one for each:
(a) Example: $a = 2$, $b = 4$, $c = 6$.
(b) Example: $p = 2$, $q = 3$, then $p + q = 5$ is prime.
(c) Example: $n = 11$, then $n^2 + n + 11 = 11 \cdot 13$ is not prime.
(d) Example: $a = 12$, $b = 11$, then $a^2 - b^2 = 144 - 121 = 23$.
(e) Example: $\sqrt{2} + (-\sqrt{2}) = 0$ is rational, but $\sqrt{2}$ and $-\sqrt{2}$ is irrational.
(f) Example: $\sqrt{4} = 2$ is rational.

4. (a) Induct on n with base case $n = 1$. Inductive step: if $F_1 + \cdots + F_{2n+1} = F_{2n+2}$ then $F_1 + \cdots + F_{2n+1} + F_{2n+3} = F_{2n+2} + F_{2n+3} = F_{2n+4}$ as required.
(b) Clearly, if $6|n$ then $2|n$ and $3|n$. For the other direction, if $2|n$ then $n = 2k$. Then if $3|2k$ we must have $3|k$ since $3 \nmid 2$. So $k = 3a$, and thus $n = 6a$, meaning $6|n$.
(c) Induct on n with base case $n = 1$. Inductive step: If $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$, then $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}} = 2-\frac{1}{2^n}+\frac{1}{2^{n+1}} = 2-\frac{1}{2^{n+1}}$ as required.
(d) If $p|a \cdot a$ then $p|a$ or $p|a$. Since the two conclusion statements are the same, we have $p|a$.
(e) Note that $33 + 9b$ is divisible by 3 but not 9. But then a^2 is divisible by 3 (by previous part) which would mean $3|a$ and thus $9|a$, but this is impossible.
(f) If $p|k^2$ and $p|(k + 1)^2$ then by (d) we have $p|k$ and $p|(k + 1)$ so that $p|(k + 1) - k = 1$, impossible.
(g) Induct on n with base case $n = 1$. Inductive step: if $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ then $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)} = \frac{n}{n+1}+\frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ as required.
(h) First, $A \subseteq B$ because if $n = 4a+6b$ then $n = 2(2a+3c) \in B$. Also, $B \subseteq A$ because if $n = 2c$ then $n = 4(2c)+6(-c) \in A$ via Euclidean algorithm calculation.
(i) If $n = a + (a + 1) + (a + 2) + (a + 3)$ then $n = 4a + 6 \equiv 2 \pmod 4$ because $(4a + 6) - 2$ is divisible by 4.
(j) Note $\gcd(n, n + p) = \gcd(n, p)$ by gcd properties. Then $\gcd(n, p)$ divides p so is either 1 or p, and it is equal to p if and only if $p|n$ (by definition of gcd).
(k) Induct on n with base case $n = 1$. Inductive step: if $a_n = 3^n - 2$ then $a_{n+1} = 3(3^n - 2) + 4 = 3^{n+1} - 2$ as claimed.
(l) If $n \in C$, then $n = 6c$ for some c. Then $n = 10(2c) + 14(-c) \in D$ as required.
(m) Observe that $\overline{n-1}\cdot\overline{n-1} = \overline{-1}\cdot\overline{-1} = \overline{1}$ so $\overline{n-1}$ is its own multiplicative inverse mod n.
(n) Induct on n with base case $n = 1$. Inductive step: if $b_n = 2^n + n$ then $b_{n+1} = 2(2^n + n) - n + 1 = 2^{n+1} + (n + 1)$ as claimed.
(o) Strong induction on n with base case $n = 2$. Inductive step: assume every integer $1 < k < n$ has a prime divisor. If n is prime, result is immediate. If n is composite, then n has a factor $1 < a < n$. Then a has a prime divisor p with $p|a$ by hypothesis, so since $p|a$ and $a|n$, $p|n$ as required.
(p) Induct on n with base cases $n = 1$ and $n = 2$. Inductive step: if $c_n = 2^{F_n}$ and $c_{n-1} = 2^{F_{n-1}}$ then $c_{n+1} = c_n c_{n-1} = 2^{F_n}2^{F_{n-1}} = 2^{F_n+F_{n-1}} = 2^{F_{n+1}}$ as required.
(q) Induct on n with base cases $n = 1$ and $n = 2$. Inductive step: if $d_n = 2^n$ and $d_{n-1} = 2^{n-1}$ then $d_{n+1} = 2^n+2(2^{n-1}) = 2^n + 2^n = 2^{n+1}$ as required.
(r) If $a = b$ then $\gcd(a, a) = a = \operatorname{lcm}(a, a)$. Conversely if $\gcd(a, b) = \operatorname{lcm}(a, b)$ then every prime must appear to the same power in the prime factorizations of a and b (since otherwise the higher power would be the power in the lcm and the lower power would be the power in the gcd), hence $a = b$.
