

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

1. (The A-to-Zs of Counting) Find:

- (a) The number of ways of putting 4 labeled objects into 3 labeled boxes.
 - (b) The number of ways of putting 4 unlabeled objects into 3 labeled boxes.
 - (c) The number of ways of putting 4 unlabeled objects into 3 unlabeled boxes.
 - (d) The number of ways of putting 4 labeled objects into 3 unlabeled boxes.
 - (e) The coefficient of $x^{20}y^{19}$ in $(2x + 3y)^{39}$.
 - (f) The coefficient of x in $(x^2 - 2/x)^{11}$.
 - (g) The number of ordered quadruples (a, b, c, d) of nonnegative integers with $a + b + c + d = 18$.
 - (h) The number of ordered quadruples (a, b, c, d) of positive integers with $a + b + c + d = 18$.
 - (i) The number of permutations of the letters in the word HALEAKALA.
 - (j) The number of permutations of the letters in the word POTATONESS that contain the string TOTS.
 - (k) The number of five-digit strings that are palindromes (i.e., read the same forwards as backwards).
 - (l) The number of five-digit strings that contain either a 4 or a 7, or both.
 - (m) The number of five-digit strings that contain both a 4 and a 7.
 - (n) The number of five-digit strings with at least 4 of the same digit in a row.
 - (o) The number of five-digit strings that are strictly increasing from left to right (e.g., including the strings 02679 and 12568 but not 11367 or 13289).
 - (p) The number of five-digit strings that are nondecreasing from left to right (e.g., including the strings 02679 and 12568 and 11367 but not 13289).
 - (q) The number of ways to select a bouquet of 12 flowers from roses, lilies, daffodils, and chrysanthemums, if the order of the flowers is irrelevant.
 - (r) The number of possible 4-card hands from a standard 52-card deck. (Hands are unordered.)
 - (s) The number of possible 4-card hands from a standard 52-card deck, where only the suits are considered (e.g., three hearts and a club, or four spades).
 - (t) The number of possible 4-card hands from a standard 52-card deck, where only the ranks are considered (e.g., A39K or J55J).
 - (u) The number of positive integer divisors of $648000 = 2^6 3^4 5^3$.
 - (v) The number of positive integers less than or equal to 2019 that are divisible by 6 or 10.
 - (w) The number of positive integers less than or equal to 2019 that are divisible by 2, 3, or 5.
 - (x) The number of $f : \{a, b, c, d, e, f, g\} \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ that are (i) functions, (ii) one-to-one functions, (iii) onto functions.
 - (y) The number of $f : \{1, 2, 3\} \rightarrow \{a, b, c, d, e, f, g\}$ that are (i) functions, (ii) one-to-one functions, (iii) onto functions.
 - (z) The number of $f : \{a, b, c, d, e, f, g\} \rightarrow \{1, 2, 3\}$ that are (i) functions, (ii) one-to-one functions, (iii) onto functions.
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2. Suppose $m \leq k \leq n$ are integers. The goal of this problem is to prove that $\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$.

- (a) Prove the identity using the definition of binomial coefficients in terms of factorials.
 - (b) Observe that the left-hand side counts the number of ways of painting k balls magenta out of a total of n , and then painting m of those k balls turquoise. Explain why the right-hand side also counts this same quantity, and deduce that the two expressions are equal.
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3. Let m, n be positive integers and $r \leq m, n$.

- (a) Prove the Chu-Vandermonde identity: $\sum_{k=0}^r \binom{n}{k} \binom{m}{r-k} = \binom{m+n}{r}$. [Hint: How many ways are there to select r unordered balls from among n distinct navy balls and m distinct maroon balls?]
 - (b) Prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$.
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4. Let p be a prime.

- (a) Show that the binomial coefficient $\binom{p}{k}$ is divisible by p for each integer k with $0 < k < p$.
 - (b) Prove that $a^p \equiv a \pmod{p}$ for every positive integer a . [Hint: Fix p and use induction on a .]
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5. Let a be a positive integer and p be a prime. Consider the set of circular necklaces with p equally spaced beads, each of which is colored in one of a colors.

- (a) How many necklaces are there in total? (Ignore rotations and reflections.)
 - (b) Declare two necklaces to be the same if it is possible to rotate one into the other. How many different necklaces are there in total? [Hint: There are two cases: one where the beads are all the same color, and the other where they are not.]
 - (c) Show that $a^p \equiv a \pmod{p}$. [Hint: The number from part (b) is counting something, so it is an integer.]
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6. Let p be a prime and a be an integer relatively prime to p .

- (a) If S is the set of residue classes modulo p , prove that the function $f : S \rightarrow S$ given by $f(\bar{b}) = \bar{a} \cdot \bar{b}$ is a bijection. [Hint: \bar{a} has a multiplicative inverse \bar{a}^{-1} modulo p .]
 - (b) Suppose u_1, u_2, \dots, u_{p-1} represent the distinct nonzero residue classes modulo p . Show that $(au_1) \cdot (au_2) \cdot \cdots \cdot (au_{p-1}) \equiv u_1 \cdot u_2 \cdot \cdots \cdot u_{p-1} \pmod{p}$. [Hint: Use (a) to show that the two products consist of the same terms, merely permuted.]
 - (c) Prove that $a^{p-1} \equiv 1 \pmod{p}$ and deduce that $a^p \equiv a \pmod{p}$.
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Remark: Problems 4, 5, and 6 give three different proofs (all with a different flavor, but all involving counting!) of a famous result of elementary number theory called Fermat's little theorem. Compare to problem 6(d) of homework 5, which verified the calculation for $p = 5$.