E. Dummit's Math 1365 ∼ Intro to Proof, Fall 2019 ∼ Homework 8, due Nov 13th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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1. Suppose $f : A \to B$ is a function.

   - If $S \subseteq A$, we write $f(S) = \{f(s) : s \in S\}$ and call $f(S)$ the image of S.
   - If $T \subseteq B$, we write $f^{-1}(T) = \{a \in A : f(a) \in T\}$ and call $f^{-1}(T)$ the inverse image of T.
   - When $T = \{b\}$ is a single element, we write $f^{-1}(T)$ as $f^{-1}(b)$ rather than $f^{-1}(\{b\})$, with the understanding that $f^{-1}(b)$ is a set (which could be empty or contain more than one element).

   (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is the function with $f(x) = x^2$. Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, and $f^{-1}([4,9])$.

   (b) Suppose $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = \sin(x)$. Find $g^{-1}(0)$, $g^{-1}(2)$, and $g^{-1}([-1,1])$.

   (c) If S is any subset of A, show that $S \subseteq f^{-1}(f(S))$.

   (d) If T is any subset of B, show that $f(f^{-1}(T)) \subseteq T$.

   (e) If B_1 and B_2 are subsets of B, show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

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2. Consider all of the 4-letter strings that can be made from the letters ABCDEF. Determine the number of such strings satisfying the given conditions:

   (a) No conditions.

   (b) The string starts with A.

   (c) The string has no repeated letters.

   (d) The string has at least one repeated letter.

   (e) The string contains at least one B.

   (f) Each letter is different from the one before it.

   (g) The string is in alphabetical order and has no repeated letters.

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3. Let A be a set of finite cardinality n.

   (a) Determine the number of relations on A in terms of n. [Hint: How many possible ordered pairs are there?]

   (b) Determine the number of reflexive relations on A in terms of n. [Hint: What conditions on the ordered pairs are there?]

   (c) Determine the number of reflexive, symmetric relations on A in terms of n.

   (d) Determine the number of reflexive, antisymmetric relations on A in terms of n.

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4. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a function such that $f(f(n)) = -n$ for all $n \in \mathbb{Z}$.

   (a) Show that f is a bijection.

   (b) Give an explicit example of such a function f. (You don't need to give an explicit formula, but at least describe how to find the values of f.)

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5. The goal of this problem is to give an alternate proof that $\mathbb{R}$ is uncountable. Let $f : \mathcal{P}(\mathbb{Z}_{>0}) \to \mathbb{R}$ be the function defined as follows: $f(A)$ is the decimal whose nth decimal place is 1 if $n \in A$, and is 2 if $n \notin A$.

  (a) Find the decimal expansion to 10 digits of the value of f on each of these sets: (i) the set of even integers, (ii) the set of odd integers, and (iii) the set of prime numbers.

  (b) Prove that f is one-to-one. [Hint: You may use the fact that $f(A)$ has a unique decimal expansion for any set A.]

  (c) Using the fact that $\mathcal{P}(\mathbb{Z}_{>0})$ is uncountable, show that $\mathbb{R}$ is uncountable.

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6. Do:

  (a) If A is a set and there exists an onto function $f : \mathbb{Z}_{>0} \to A$, show that A is countable. [Hint: For any $a \in A$, let a_n be the smallest element of $\mathbb{Z}_{>0}$ with $f(n) = a$. Show that $g : A \to \mathbb{Z}_{>0}$ is one-to-one.]

  (b) Show that the union of two countably infinite sets is countable. [Hint: If the sets are $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$, show that $f : \mathbb{Z}_{>0} \to A \cup B$ given by $f(n) = \begin{cases} a_k & \text{if } n = 2k \\ b_k & \text{if } n = 2k - 1 \end{cases}$ is onto.]

  (c) Show that the set of irrational numbers is uncountable.

  (d) Show that a countable union of countable sets is countable. [Hint: If any set is empty, ignore it. Then write the sets as $S_1, S_2, S_3, \dots$ and consider the map $f : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \cup_{n=1}^{\infty} S_n$ where $f(a, b)$ is the ath element in S_b, repeating elements if needed.]

  (e) Show that the set of real numbers that can be described with a finite number of English words is countable. Deduce that there are uncountably many real numbers that cannot be described with a finite number of English words.

  • Remark: The same argument used in part (e) can also be used to show that the set of "computable numbers" (real numbers that can be computed to arbitrarily good accuracy by a finite, terminating algorithm) is also countable, and thus there are uncountably many uncomputable numbers. A similar technique can be used to show that there are uncountably many transcendental numbers.

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7. The goal of this problem is to show that there exists a bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$.

  (a) Show that there exists a one-to-one map $f : [0, 1] \to [0, 1] \times [0, 1]$.

  (b) Consider the map $g : [0, 1] \times [0, 1] \to [0, 1]$ where $g(0.d_1 d_2 d_3 \dots, 0.e_1 e_2 e_3 \dots) = 0.1 d_1 e_1 1 d_2 e_2 1 d_3 e_3 \dots$, where we always choose the decimal expansion ending in a string of 9s if there is a choice. Show that g is one-to-one.

  (c) Deduce that there exists a bijection between $[0, 1]$ and $[0, 1] \times [0, 1]$.

  (d) Show that there exists a bijection between $[0, 1]$ and $\mathbb{R}$. Deduce that there exists a bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$. [Hint: Use $f(x) = x$ and $g(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$.]

  • Remark: Some other (surprising!) results related to these are (i) there exists a *continuous* onto function $f : [0, 1] \to [0, 1] \times [0, 1]$, but (ii) there does not exist a continuous bijection $f : [0, 1] \to [0, 1] \times [0, 1]$. Functions with the property (i) are often called space-filling curves.