

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

1. For each relation R on A , identify whether or not R is (i) a partial ordering, and (ii) a total ordering on A .

- (a) $A = \{a, b, c\}$, $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$.
 - (b) $A = \{4, 6, 8, 10, 12\}$ and $R = \{(4, 4), (4, 8), (4, 12), (6, 6), (6, 12), (8, 8), (10, 10), (12, 12)\}$ is the divisibility relation on A .
 - (c) $A = \mathbb{R}$, $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 \leq y^2\}$.
 - (d) $A = \mathbb{R}$, $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \leq x\}$.
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2. For each f , A , and B , identify whether or not f is a function from A to B , and if not, explain why not.

- (a) $A = \{1, 2, 3\}$, $B = \{4\}$, $f = \{(1, 4), (2, 4), (3, 4)\}$.
 - (b) $A = \{1\}$, $B = \{2, 3, 4\}$, $f = \{(1, 2), (1, 3), (1, 4)\}$.
 - (c) $A = \{1, 2, 3\}$, $B = \{4\}$, $f = \{(1, 2), (2, 3), (3, 4)\}$.
 - (d) $A = \mathbb{Q}$, $B = \mathbb{Q}$, where $f(a/b) = a/b^2$.
 - (e) $A = \mathbb{Q}$, $B = \mathbb{Q}$, where $f(a/b) = a^2/b^2$.
 - (f) $A = \mathbb{R}$, $B = \mathbb{Z}$, with $f = \{(x, n) \in \mathbb{R} \times \mathbb{Z} : n \leq x < n + 1\}$.
 - (g) $A = \mathbb{Z}$, $B = \mathbb{Z}/m\mathbb{Z}$, where $f(a) = \bar{a}$, with m a fixed positive integer.
 - (h) $A = \mathbb{Z}/m\mathbb{Z}$, $B = \mathbb{Z}$, where $f(\bar{a}) = a$, with m a fixed positive integer.
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3. For each function $f : A \rightarrow B$, determine whether f is (i) one-to-one, (ii) onto, and (iii) a bijection.

- (a) $f_1(x) = 2x + 1$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
 - (b) $f_2(n) = 2n + 1$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.
 - (c) $f_3(x) = x^2$ from $A = \mathbb{R}_+$ to $B = \mathbb{R}_+$.
 - (d) $f_4(x) = \frac{2x - 1}{x + 3}$ from $A = \mathbb{R} \setminus \{-3\}$ to $B = \mathbb{R}$.
 - (e) $f_5(n) = \frac{1}{n^2 + 1}$ from $A = \mathbb{Z}$ to $B = \mathbb{Q}$.
 - (f) $A = \mathbb{Z}$, $B = \mathbb{Z}/m\mathbb{Z}$, where $f(a) = \bar{a}$, with m a fixed positive integer.
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4. Suppose A , B , and C are sets.

- (a) If $f : B \rightarrow C$ and $g : A \rightarrow B$ are both one-to-one, prove that $f \circ g$ is also one-to-one.
 - (b) If $f : B \rightarrow C$ and $g : A \rightarrow B$ are both onto, prove that $f \circ g$ is also onto.
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5. Suppose $f : A \rightarrow B$ is a function, and S is an equivalence relation on B . Prove that the relation $R : A \rightarrow A$ given by $R = \{(a, b) \in A \times A : (f(a), f(b)) \in S\}$ is an equivalence relation on A .

6. Let A be a set containing exactly n elements for a positive integer n , and let \leq be a total ordering on A .
- Prove that A contains a greatest element, which is to say, an element x such that $a \leq x$ for all $a \in A$. [Hint: Use strong induction. For the inductive step, choose two nonempty proper subsets of A whose union is A , and compare the greatest elements of each.]
 - Prove that it is possible to label the elements of A as $\{a_1, a_2, \dots, a_n\}$ such that $a_i \leq a_j$ precisely when $i \leq j$. [Hint: Use (a) to identify a greatest element, and then apply induction to the rest of A .]
 - Demonstrate the result of part (b) by identifying the labels a_i for the divisibility ordering on the set $\{2, 12, 6, 36, 144\}$.
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7. The goal of this problem is to outline the construction of the rational numbers \mathbb{Q} using equivalence classes. Let $S = \mathbb{Z} \times \mathbb{Z}_{\neq 0}$ be the set of ordered pairs (a, b) of integers with $b \neq 0$. Define the relation \sim by saying $(a, b) \sim (c, d)$ precisely when $ad = bc$.

- Prove that \sim is an equivalence relation on S . [Hint: For transitivity, use the fact that $f(ad - bc) + b(cf - de) = d(af - be)$.]

If a, b are integers with $b \neq 0$, we now think of $\frac{a}{b}$ as being the equivalence class $[(a, b)]$. Motivated by the usual addition and multiplication of fractions as $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, we also define addition and multiplication on equivalence classes by $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$ and $[(a, b)] \cdot [(c, d)] = [(ac, bd)]$.

- Show that multiplication is well-defined, in the sense that if $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a'}{b'} \cdot \frac{c'}{d'}$. In other words, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, show that $(ac, bd) \sim (a'c', b'd')$.
 - Show that addition is also well-defined.
 - Show that $[(0, 1)]$ is an additive identity and $[(1, 1)]$ is a multiplicative identity, namely, that $[(a, b)] + [(0, 1)] = [(a, b)]$ and $[(a, b)] \cdot [(1, 1)] = [(a, b)]$.
 - Show that the element $[(a, b)]$ has an additive inverse $[(-a, b)]$ and if $a \neq 0$, a multiplicative inverse $[(b, a)]$. In other words, show that $[(a, b)] + [(-a, b)] = [(0, 1)]$ and $[(a, b)] \cdot [(b, a)] = [(1, 1)]$.
 - [Optional] Show that addition and multiplication are commutative and associative, and that multiplication distributes over addition.
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