E. Dummit's Math 1365 ~ Intro to Proof, Fall 2019 ~ Homework 6, due Oct 28th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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1. (Midterm 1, Echo) Prove that the integers a and b are relatively prime if and only if there exists no prime p such that $p|a$ and $p|b$. [Hint: Prove the contrapositive.]

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2. For each of the following relations, decide whether they are (i) symmetric, (ii) transitive, (iii) reflexive, and (iv) an equivalence relation. (Justification is not required.)

   (a) $R_1 = \{(1,1),\ (2,1),\ (2,2)\}$ on the set $\{1,2\}$.

   (b) $R_2 = \{(1,1),\ (2,1),\ (2,2)\}$ on the set $\{1,2,3\}$.

   (c) $R_3 = \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (3,1),\ (3,2)\}$ on the set $\{1,2,3\}$.

   (d) $R_4 = \{(1,1),\ (2,2),\ (2,3),\ (3,2),\ (3,3)\}$ on the set $\{1,2,3\}$.

   (e) R_5, the relation on human beings where $a\ R_5\ b$ means "a has the same last name as b".

   (f) R_6, the relation on human beings where $a\ R_6\ b$ means "a is a parent of b".

   (g) R_7, the empty relation on the empty set. (Be very careful with the quantifiers in the definitions of the terms!)

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3. Suppose $\mathcal{F}$ is a family of sets. For each relation defined on $\mathcal{F} \times \mathcal{F}$ below, give a simple description of the relation in words:

   (a) $R_1 = \{(A,B)\ :\ \exists x \in A,\ x \in B\}$.

   (b) $R_2 = \{(A,B)\ :\ \forall x \in A,\ x \in B\}$.

   (c) $R_3 = \{(A,B)\ :\ \forall x,\ (x \in A) \Leftrightarrow (x \in B)\}$.

   (d) $R_4 = \{(A,B)\ :\ \forall x \in C,\ (x \in A) \vee (x \in B)\}$, where C is a fixed set.

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4. Find all partitions of the set $\{1,2,3\}$ and write down the corresponding equivalence relation for each.

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5. Suppose that R is a relation on the set A.

   Proposition: If R is symmetric and transitive, then R is reflexive.
   Proof: Let $a \in A$ be arbitrary. Because R is symmetric, if $a\ R\ b$ then $b\ R\ a$. Therefore, applying transitivity to $a\ R\ b$ and $b\ R\ a$ yields $a\ R\ a$. Because a was arbitrary, we conclude $a\ R\ a$ for every $a \in A$, so R is reflexive.

   (a) The proof given above is erroneous. (If it were correct, we would not bother to include reflexivity in the definition of an equivalence relation!) Explain, briefly, what the error in the proof is. [Hint: See problem 5(b) of homework 2.]

   (b) Construct a counterexample to the proposition using the set $A = \{1,2\}$.

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6. Suppose $R : A \to B$ and $S : A \to B$ are relations (i.e., subsets of $A \times B$). For each statement below, identify whether it is true or false. If it is true then prove it, and if it is false then give a counterexample.

   (a) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

   (b) $(R \cup S)^{-1} = R^{-1} \cap S^{-1}$.

   (c) $R = R^{-1}$ if and only if R is symmetric.

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7. There are a few other properties of relations that arise from time to time. We say a relation R on a set A is <u>irreflexive</u> when $a \not R\, a$ for all $a \in A$ (this is essentially the opposite of being reflexive), and we say R is <u>antisymmetric</u> when $a\ R\ b$ and $b\ R\ a$ together imply that $a = b$ (this is essentially the opposite of being symmetric).

   Examples: The order relation $<$ on real numbers is irreflexive, because $a < a$ is false for all real numbers a. The order relation $\leq$ on real numbers is antisymmetric, because $a \leq b$ and $b \leq a$ implies $a = b$.

   (a) Show that the divisibility relation $\mid$ on positive integers is antisymmetric, but the divisibility relation $\mid$ on all integers is *not* antisymmetric.

   (b) For each relation in problem 2, determine whether it is (v) irreflexive, and (vi) antisymmetric.

   (c) Show that "R is not reflexive" is not the same as "R is irreflexive", and also that "R is not symmetric" is not the same as "R is antisymmetric".

   (d) Prove that the only relations on A that are both symmetric and antisymmetric are the subsets of the identity relation.

   (e) Does there exist a relation on $A = \{1, 2, 3\}$ that is both reflexive and irreflexive? Does there exist any relation on any set that is both reflexive and irreflexive? Explain why or why not.
