

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

1. Let $U = \mathbb{R}_{\geq 0}$, the nonnegative real numbers, and take $P(x, y)$ to be the statement " $x < y$ ".
 - (a) There are 8 possible ways, listed below, of quantifying both x and y in the statement $P(x, y)$. For each statement, translate it into words and then find its truth value.
 - i. $\forall x \forall y, P(x, y)$
 - ii. $\forall x \exists y, P(x, y)$
 - iii. $\exists x \forall y, P(x, y)$
 - iv. $\exists x \exists y, P(x, y)$
 - v. $\forall y \forall x, P(x, y)$
 - vi. $\forall y \exists x, P(x, y)$
 - vii. $\exists y \forall x, P(x, y)$
 - viii. $\exists y \exists x, P(x, y)$
 - (b) Repeat part (a) with $P(x, y)$ replaced by the statement " $y = 2x$ " and with universe \mathbb{Z} .
 - (c) Of the eight quantified statements in part (a), two pairs will always be logically equivalent for any statement $P(x, y)$. Identify these pairs.
-

2. For each statement, translate it into words and then find its truth value. (Assume that all capital-letter variables refer to sets.)
 - (a) $\forall x \in \mathbb{R}, x^2 > 0$.
 - (b) $\exists x \in \mathbb{Z}, x^2 - 3x + 2 = 0$.
 - (c) $\forall A \forall B \forall C, [x \in A \cap B \cap C] \Rightarrow [x \in A \cap B] \wedge [x \in A \cup C]$.
 - (d) $\forall A \exists x (x \in A)$.
 - (e) $\forall A \exists x \exists y, (A = \emptyset) \vee [(x \in A) \wedge (y \in A)]$.
-

3. Negate each given statement and then rewrite the result as an equivalent positive statement. (All quantifiers should appear ahead of any negation operators.)
 - (a) $\exists x, x^2 = 2$.
 - (b) $\exists x \exists y, x + y \neq 5$.
 - (c) $\forall x \exists y \exists z, x \cdot y + z > 2$.
 - (d) $\forall a \in A \exists b \in B (a \in C \wedge b \in C)$.
-

4. The goal of this problem is to describe a way to define the intersection and union of two sets using only the subset relation. Let A and B be any sets.

- (a) Suppose $C \subseteq A$ and $C \subseteq B$. Show that $C \subseteq A \cap B$.
- (b) Suppose X is a set such that $X \subseteq A$ and $X \subseteq B$, and also X contains (as a subset) every set C with $C \subseteq A$ and $C \subseteq B$. Prove that $X = A \cap B$. [Hint: Show separately that $X \subseteq A \cap B$ and that $A \cap B \subseteq X$.]
- (c) Deduce that $[X = A \cap B] \Leftrightarrow (X \subseteq A) \wedge (X \subseteq B) \wedge \forall C, [(C \subseteq A) \wedge (C \subseteq B) \Rightarrow C \subseteq X]$.
- (d) Suppose $A \subseteq D$ and $B \subseteq D$. Show that $A \cup B \subseteq D$.
- (e) Suppose Y is a set such that $A \subseteq Y$ and $B \subseteq Y$, and also Y is contained (as a subset) in every set D with $A \subseteq D$ and $B \subseteq D$. Prove that $Y = A \cup B$.
- (f) Deduce that $[Y = A \cup B] \Leftrightarrow (A \subseteq Y) \wedge (B \subseteq Y) \wedge \forall D, [(A \subseteq D) \wedge (B \subseteq D) \Rightarrow Y \subseteq D]$.

- **Remark:** The result of part (b) is known as the maximality property of the intersection, and provides a way to define the intersection of two sets using only the subset relation (i.e., without speaking about specific elements). Part (e) establishes the analogous minimality property of the union.
-

5. The goal of this problem is to examine the quantifier “there exists a unique”, written as $\exists!$. Thus, for example, the statement “there exists a unique x such that $x^2 = 2$ ” would be written $\exists!x, x^2 = 2$. The meaning of this quantifier is that there exists an element x satisfying the hypotheses, and that there is exactly one such x .

- (a) Identify the truth values of the following statements:
 - i. $\exists!n \in \mathbb{Z}, n^2 = 2$.
 - ii. $\exists!n \in \mathbb{Z}, n^2 = 4$.
 - iii. $\exists!n \in \mathbb{Z}_+, n^2 = 4$.
 - iv. $\forall x > 0 \exists!y > 0, xy = 1$, with universe \mathbb{R} .
 - (b) It may seem that $\exists!$ is a new quantifier, but in fact, it can be expressed in terms of \exists and \forall . Explain why $\exists!x \in A, P(x)$ is logically equivalent to $\exists x \in A, P(x) \wedge [\forall y \in A, P(y) \Rightarrow (y = x)]$ for any proposition $P(x)$. (Your explanation does not have to be fully rigorous.)
-

6. Show the following:

- (a) For all positive integers n , show that the sum $1^2 + 2^2 + 3^2 + \dots + n^2$ equals $\frac{n(n+1)(2n+1)}{6}$.
 - (b) For all positive integers n , show that the sum $3^0 + 3^1 + 3^2 + \dots + 3^n$ equals $\frac{3^{n+1} - 1}{2}$.
-