E. Dummit's Math 1365 $\sim$ Intro to Proof, Fall 2019, $\sim$ Homework 2, due Sep 23rd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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1. Suppose $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 5, 7, 8\}$, and $C = \{2, 3, 5, 7\}$, with universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Calculate the following (no justification is required):
   (a) $A \cap B$.
   (b) $A \cup C$.
   (c) $B^c \cap C^c$.
   (d) $(A^c \cup B) \cap (B^c \cup C)$.

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2. Suppose $A = \{1, 2, \{1\}, \emptyset\}$. Identify each of the statements below as true or as false, and give a brief (1-sentence) explanation why.
   (a) $\emptyset \in A$.
   (b) $\emptyset \subseteq A$.
   (c) $\{1\} \in A$.
   (d) $\{1\} \subseteq A$.
   (e) $\{1, 2\} \in A$.
   (f) $\{1, 2\} \subseteq A$.
   (g) $\{\{1\}\} \in A$.
   (h) $\{\{1\}\} \subseteq A$.

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3. In a certain intro to proofs course, there are 90 students enrolled. Of these, 67 do their homework, 62 attend class, and 61 are receiving an A. There are 54 students who do their homework and are receiving an A, and 49 of these also attend class. One very motivated student is receiving an A without doing homework or attending class, and every student who is both attending class and also doing their homework is getting an A.
   (a) How many students are getting an A and doing their homework, but not going to class?
   (b) How many students are going to class, but not doing their homework and getting an A?
   (c) How many students are going to class or doing their homework?
   (d) How many students are not getting an A, not going to class, and not doing their homework?
   - Remark: The moral of this problem is that to get an A, you should do your homework and go to class!

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4. Suppose A, B, and C are any sets contained in a universal set U.
   (a) Prove that $A \subseteq B$ if and only if $A \cup B = B$.
   (b) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (c) Prove that $(A \cup B)^c = A^c \cap B^c$.
   (d) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   - Remark: Note that each of these parts asks for a proof of a result stated (but not explicitly proven) in the course notes. As such, you may not quote those results in the responses (e.g., your response to part (b) cannot be "This follows immediately from the distributive laws for unions and intersections"), because it would be circular logic.

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5. Suppose A and B are sets. The goal of this problem is to study the question of when $A \times B = B \times A$.

Proposition: $A \times B = B \times A$ if and only if $A = B$.
Proof: If $A = B$, then clearly $A \times B = A \times A = B \times A$. Now suppose $A \times B = B \times A$, and let $a \in A$ and $b \in B$ be arbitrary elements of A and B respectively. Then by definition, $(a, b) \in A \times B$, and so by hypothesis, $(a, b) \in B \times A$. This means $a \in B$ and $b \in A$. Since $a \in A$ and $b \in B$ are arbitrary, the fact that $a \in B$ implies $A \subseteq B$, and the fact that $b \in A$ implies $B \subseteq A$. We conclude that $A = B$, as required.

(a) Consider the proposition and proof given above. Show that the proposition is incorrect by explaining why taking $A = \emptyset$ and $B = \{1, 2\}$ yields a counterexample.

(b) Part (a) shows that the proposition stated above is incorrect, so the proof must contain a logical error. Identify what the error is, and why it causes the proof to be incorrect. [Hint: The counterexample from part (a) is clearly relevant.]

(c) Give a corrected version of the proposition, and then give a correct proof. [Hint: Your corrected proposition should start with "$A \times B = B \times A$ if and only if $A = B$ or..."]

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6. In addition to union and intersection, there are a few other set operations that arise from time to time. Two of these are the set difference $A \backslash B = \{x \in A : x \notin B\}$, the set of elements of A not in B, and the symmetric difference $A \Delta B = (A \backslash B) \cup (B \backslash A)$, the elements in either A or B but not both. (Observe that $A \Delta B = B \Delta A$, whence the name *symmetric* difference.)

(a) If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$, and $C = \{1, 3, 5\}$, find $A \backslash B$, $B \backslash A$, $A \backslash C$, $C \backslash A$, $B \backslash C$, and $C \backslash B$.

(b) If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$, and $C = \{1, 3, 5\}$, find $A \Delta B$, $A \Delta C$, and $B \Delta C$.

(c) If A and B are subsets of a universal set U, show that $A \backslash B = A \cap B^c$. Deduce that $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$.

(d) Using a Venn diagram or otherwise, decide which of the following statements are true and which are false (you do NOT need to prove the true statements nor give counterexamples to the false ones):

i. $(A \backslash B) \cup B = A$.
ii. $(A \backslash B) \cup (A \backslash B^c) = A$.
iii. $A \Delta B = (A \cup B) \backslash (A \cap B)$.
iv. $(A \backslash B) \cap (B \backslash A) = \emptyset$.
v. $(A \Delta B)^c = A^c \backslash B$.
vi. $(A \backslash B) \backslash C = (A \backslash C) \backslash B$.
vii. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.
viii. $(A \cap B) \Delta C = (A \Delta C) \Delta (A \backslash B)$.

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7. Wikipedia has many articles. Some of these articles are lists, such as the article "List of American mathematicians". There are even articles which list other lists, such as the article "List of lists of mathematical topics". Some of these lists of lists contain themselves, such as the article "List of lists of lists".

(a) Consider the Wikipedia article titled "List of lists that do not contain themselves". Can this article be listed on itself? (Assume, as most students do, that everything on Wikipedia is accurate.)

(b) The only true test of a hypothesis is empirical evidence. To this end, what *actually happens* when you visit the Wikipedia article "List of lists that do not contain themselves"?
