E. Dummit's Math 1365 ∼ Intro to Proof, Fall 2019 ∼ Homework 10, due Dec 4th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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1. Find:
   (a) The coefficient of $a^2b^3c^4$ in the expansion of $(a+b+c)^9$.
   (b) The coefficient of $a^2bc^2d^2$ in the expansion of $(a+2b+3c+4d)^7$.
   (c) The number of different monomial terms in the expansion of $(a+b+c+d+e)^{11}$.
   (d) The number of zeroes at the end of 777! when written in base 10.
   (e) The number of zeroes at the end of 777! when written in base 12.
   (f) The number of zeroes at the end of 777! when written in base 777.
   (g) The number of different ways that the 7 students in a class can peer-grade one another's homework so that each person grades one assignment, and no student grades their own assignment.
   (h) The number of reflexive, symmetric, transitive relations on the set $\{1,2,3,4,5,6,7\}$.

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2. A "lyrical pattern" consists of a sequence of long and short beats, where a long beat is twice as long as a short beat. Some examples are long-long-short-long (length 7) and short-short-short-short-long (length 6). Prove the number L_n of lyrical patterns whose length equals n short beats equals the Fibonacci number F_{n+1}. [Hint: The last beat is either short or long. What happens if that beat is deleted?]
   - Remark: The study of such patterns by Indian poets writing in Sanskrit (e.g., Virahanka in approximately the year 700 CE) is the first known analysis of the Fibonacci numbers.

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3. Suppose m and n are positive integers. Show that $\dfrac{(m+n)!}{(m+n)^{m+n}} < \dfrac{m!}{m^m} \cdot \dfrac{n!}{n^n}$. [Hint: Rearrange the inequality into $\square < (m+n)^{m+n}$.]

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4. Solve the following:
   (a) Show that if 51 elements from the set $\{1,2,\dots,100\}$ are selected, then at least one pair of consecutive integers must be among the chosen 51 elements.
   (b) For fall 2019 semester classes running on a normal Mon/Wed/Thu schedule, there are a total of 36 lecture days. Suppose that a student is taking 5 such courses, each of which has 2 midterm examinations. Assuming that no course schedules an examination before lecture day 7 and no course schedules an examination after lecture day 33, show that the student must have at least one pair of exams with at most 1 lecture day in between.
   (c) Show that if 10 points are chosen inside a square of side length 3, at least one pair of points must be within a distance $\sqrt{2}$ of one another.
   (d) Suppose five points are drawn on the surface of a sphere. Prove that at least four of the points must lie on some closed hemisphere. [Hint: Choose two points and cut the sphere along the great circle that contains them. Where can the other three points be?]
   (e) Show that if 100 distinct integers are chosen, there exists a subset S of 10 of these integers such that the difference between any pair of integers in S is divisible by 11.
   (f) Let A be a set of n integers. Prove that A contains a nonempty subset whose sum of elements is divisible by n. [Hint: Consider the sums a_1, a_1+a_2, $a_1+a_2+a_3$, ... , $a_1+\cdots+a_n$ modulo n. If two of them are the same, subtract them.]

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5. The goal of this problem is to prove that the Fibonacci numbers are periodic modulo m for any modulus $m > 1$. So let F_n be the nth Fibonacci number.

   (a) Suppose $F_{k+a} \equiv F_k$ modulo m and $F_{k+a+1} \equiv F_{k+1}$ modulo m. Prove that $F_{n+a} \equiv F_n$ modulo m for all integers n. [Hint: Use strong induction on n. Make sure to induct upwards for all $n \geq k$ and also induct downwards for all $n \leq k+1$.]

   (b) Prove that the Fibonacci numbers are periodic modulo m, which is to say, there exists a positive integer a such that $F_{n+a} \equiv F_n$ modulo m for all integers n. [Hint: There are only finitely many possible pairs (F_p, F_{p+1}) modulo m.]

   (c) Illustrate part (b) by finding the period of the Fibonacci numbers modulo 2, modulo 3, and modulo 5.

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6. Each of the balls in a collection of orange balls and purple balls is labeled with an integer between 1 and 60 inclusive, and no two balls of the same color receive the same labeling.

   (a) If there are 11 balls of each color, show that there are at least two pairs of differently-colored balls whose labels have the same total. (For example: 11 orange + 37 purple = 28 orange + 2 purple.)

   (b) What if there are only 9 balls of each color? [Hint: Try taking the orange balls to be numbered $\{1, 2, 3, 4, 5, 56, 57, 58, 59\}$ and see what labels could work for the purple ones to avoid duplicate pairs.]

   - Remark: For a more serious challenge, try to figure out what the answer might look like in general, where the labels can range from 1 to n inclusive. This particular question is asking about the how large the size of the "sumset" $A + B = \{a + b : a \in A, b \in B\}$ can be in terms of the sizes of the original sets A and B. Problems of this general type are an active area of mathematical research in analytic and algebraic combinatorics.

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7. [Optional] If $\alpha \in \mathbb{R}$, we may uniquely write $\alpha = n + x$ where n is the greatest integer less than or equal to α, and $0 \leq x < 1$: we call x the fractional part of α and write $x = \{\alpha\}$.

   Example: We have $\{2.4\} = 0.4$, $\{\pi\} = \pi - 3$, and $\{-0.4\} = 0.6$.

   (a) If m_1 and m_2 are distinct integers such that $\{m_1\alpha\} = \{m_2\alpha\}$, show that α must be rational. Deduce that if α is irrational, then all of the numbers $\{m\alpha\}$ for $m \in \mathbb{Z}$ are different from one another.

   (b) Suppose α is irrational. Show that for any positive integer n, there exist integers m_1 and m_2 such that $0 < \{m_1\alpha\} - \{m_2\alpha\} < \dfrac{1}{n}$. [Hint: Consider the n intervals $[0, \frac{1}{n})$, $[\frac{1}{n}, \frac{2}{n})$, ... , $[\frac{n-1}{n}, 1)$ and note that part (a) implies that there are infinitely many different numbers of the form $\{m\alpha\}$.]

   (c) Suppose α is irrational. Show that for any positive integer n, there exists an integer m such that $0 < \{m\alpha\} < \dfrac{1}{n}$. [Hint: Write $m = m_1 - m_2$.]

   - Remark: By looking at multiples of the number $m\alpha$ from part (c), one can deduce that there is a number $\{k\alpha\}$ in any interval of width $\dfrac{1}{n}$ inside $(0, 1)$. By taking n small enough, this implies that *any* interval (x, y) inside $(0, 1)$ will contain numbers of the form $\{m\alpha\}$, no matter how small the interval is. This means that the fractional parts of the multiples of α eventually "spread out" over the entire interval $(0, 1)$.

   - Remark: Arguments and results of this nature are moderately common in mathematical analysis.
