E. Dummit's Math 1365 ∼ Intro to Proof, Fall 2019 ∼ Homework 1, due Sep 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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1. Each item below contains a proposition and a claimed proof of the proposition. Decide whether each proof is correct, and for the incorrect proofs, identify (briefly!) at least one mistake or error.

   (a) Proposition: The integer 2 is odd.
   Proof: It is a fact about odd integers that any odd integer plus any odd integer always gives an even integer. Because $2 + 2 = 4$, and 4 is an even integer, this means 2 must be odd.

   (b) Proposition: If x is an integer and $3x - 2 = 7$, then $x = 3$.
   Proof: Suppose $x = 3$. Then $3x - 2 = 3(3) - 2 = 7$. Therefore, if $3x - 2 = 7$, then $x = 3$.

   (c) Proposition: Every odd number greater than 1, except for 9, is prime.
   Proof: Clearly, 3 is prime, 5 is prime, 7 is prime, 9 is not prime, 11 is prime, and 13 is prime. Since we have excluded 9, all odd numbers greater than 1 are prime.

   (d) Proposition: If x is an integer with $x \neq 1$ and $\dfrac{x^2 - 1}{x - 1} = 3$, then $x = 2$.
   Proof: If $x \neq 1$, then $\dfrac{x^2 - 1}{x - 1} = \dfrac{(x - 1)(x + 1)}{x - 1} = x + 1$, so $\dfrac{x^2 - 1}{x - 1} = 3$ implies $x + 1 = 3$. By subtracting 1 from both sides of the equality, we conclude that $x = 2$.

   (e) Proposition: For an integer m, m is even if and only if m^2 is even.
   Proof: Suppose m is even. Then $m = 2k$ for some integer k, meaning that $m^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$ and thus m^2 is even. Therefore, m is even if and only if m^2 is even.

   (f) Proposition: Suppose that x and y are integers and $x \neq 2$. If $x^2 y = 4y$, then $y = 0$.
   Proof: Suppose $x^2 y = 4y$. Then $(x^2 - 4)y = 0$. Since $x \neq 2$, that means $x^2 \neq 4$, so $x^2 - 4 \neq 0$. Then we can cancel the nonzero term $x^2 - 4$ from $(x^2 - 4)y = 0$ to conclude that $y = 0$. Therefore, if $x^2 y = 4y$, then $y = 0$.

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2. Find a counterexample to the following statement: if p is a prime number, then $2^p - 1$ is also a prime number.

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3. After discussing Euclid's proof that there are infinitely many primes, it is frequently claimed (occasionally in actual textbooks) that if $p_1, p_2, \dots, p_k$ is a list of the first k primes, then the number $p_1 p_2 \cdots p_k + 1$ is always prime. Show that this statement is false by giving an explicit counterexample.

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4. Write explicitly the converse, inverse, and contrapositive of the following conditional statements:

   (a) If you do not study for your exams, then you will get bad grades.

   (b) If you want to bake a cake, then you must have eggs and flour.

   (c) If n is an odd integer greater than 7, then n is the sum of three odd primes.

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5. Using a truth table or otherwise, determine whether each of following pairs of statements are equivalent. For those that are false, give an explicit counterexample (i.e., truth values for the propositions showing the statements are different):

   (a) $A \wedge (A \vee B)$ and A.

   (b) $\neg(A \vee \neg B) \Rightarrow \neg B$ and $B \Rightarrow A$.

   (c) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ and $P \Rightarrow (P \Leftrightarrow Q)$.

   (d) $\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R)$ and True.

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6. Suppose that P, Q, and R are any propositions.

   (a) Prove that if $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is also true. [Hint: Try using a truth table to identify all cases in which both $P \Rightarrow Q$ and $Q \Rightarrow R$ are true.]

   (b) Suppose that the statements "If it is raining, then it is cloudy" and "If it is cloudy, then people want to stay home" are both true. Is the statement "If it is raining, then people want to stay home" necessarily true? Explain.
