MATH 1365 **Final Exam Practice Problems** Fall 2019

1. For each of the following statements, write its negation and its contrapositive.
   a) If $2 \nmid x$, then $2|(x+1)$.
   b) If an integer p is prime, then either $2^p - 1$ is prime or p is divisible by 3.
   c) Let a and b be integers. If $a(b^2 - 2b)$ is odd, then a and b are both odd.
   d) For all integers a, b, c, if $a|b$ and $b|c$, then $a|c$.

2. Show that $(\neg p) \wedge (q \vee (\neg p))$ is logically equivalent to $p \to (\neg(q \vee p))$.

3. Prove that for all integers a, b, c, if $a|b$ and $b|c$, then $a|c$.

4. Prove that there are no integers a and b such that $3a - 9b = 2$.

5. Prove the following proposition:
   If a and b are real numbers and $a + b = 0$, then $a \leq 0$ or $b \leq 0$.

6. Prove the following proposition by contrapositive.
   Let a be a positive integer. If $a \equiv 2\ (mod\ 4)$ or $n \equiv 3\ (mod\ 4)$, then a is not a perfect square.

7. Prove that for all prime p, $\sqrt{p}$ is irrational. You may use the following proposition:
   Proposition: For any prime p and integers a, b, if p is $p|ab$, then $p|a$ or $p|b$.

8. Given the sets,

$$A = \{1,2,3,4,5\}, \qquad B = \{x \in A : x \text{ is prime}\}, \qquad C = \{x \in A : x \text{ is odd}\},$$
$$D = \{x \in A : 2|x\}, \qquad E = \{x \in A : x|2\}$$

   List the elements of:
   a) $A - (B \cap C)$
   b) $(A - B) \cap (A - C)$
   c) $A - (D \cup E)$
   d) $(A - D) \cup (A - E)$
   e) $(B - C) \times (C - B)$
   f) $B \Delta C$
   g) $2^{B \cap C}$
   h) $2^B \cap 2^C$
   i) $2^{D \cup E}$
   j) $2^D \cup 2^E$

9. Prove or disprove the following statement:
   Let A, B, C be sets. If $A - B = A - C$, then $B = C$.

10. Let $A = \{x + 3y : x, y \in \mathbb{Z}\}$ and B be the set of all even integers. Prove or disprove:
    a) $A \subseteq B$
    b) $B \subseteq A$

11. Prove or disprove the following statement:
    Let p, q be distinct primes. Suppose A is the set of multiples of p, B is the set of multiples of q, C is the set of multiples of pq, then $C = A \cap B$.
    You may use the following proposition:
    Proposition: For any prime p and integers a, b, if p is $p|ab$, then $p|a$ or $p|b$.

12. Let A be a finite set and let R be a relation on A. Prove the following.
    a) $(R^{-1})^{-1} = R$
    b) $R = R^{-1}$ if and only if R is symmetric.

13. Let $A = \{1,2,3,4\}$, and consider the relation on A defined by

$$R = \{(1,1), (2,1), (3,1), (4,1), (2,3), (4,4)\}$$

Prove or disprove the following.

a) R is a function.
b) R is reflexive.
c) R is irreflexive.
d) R is symmetric.
e) R is antisymmetric.
f) R is transitive.
g) R is an equivalence relation.

14. Let $A = \{1,2,3,4\}$ and R be a relation on A.

a) Find the smallest R which is reflexive.
b) Find the smallest R which is irreflexive.
c) Explain why there is NO relation R which is both reflexive and irreflexive. Can a relation (on any set) ever be both reflexive and irreflexive?
d) Find a relation R which is neither reflexive nor irreflexive.
e) Find a relation R which is reflexive and symmetric but not transitive.
f) Find a relation R which is reflexive and transitive but not symmetric.
g) Find a relation R which is symmetric and transitive but not reflexive.
h) Let R be a relation on A defined by $R = \{(1,1), (1,2), (1,3)\}$. Find the smallest relation S on A such that $R \subset S$ and S is symmetric.
i) Let R be a relation on A defined by $R = \{(1,4), (4,1)\}$. Find the smallest relation S on A such that $R \subset S$ and S is transitive.

15. Suppose R is an equivalence relation on a set A. Show that:

For every $a, b \in A$, $a \in [b] \Leftrightarrow b \in [a]$.

16. For each of the following relations, explain why it is NOT an equivalence relation.

a) "$\subseteq$", i.e. inclusion on sets
b) "|", i.e. divides on integers
c) R is a relation on $\mathbb{Z}$, and $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y = 2x\}$
d) R is a relation on $\mathbb{Z}$, and $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x - y \geq 0\}$
e) R is a relation on $\{a,b,c\}$, and $R = \{(a,a), (b,b), (c,c), (a,c), (c,a), (b,c), (c,b)\}$

17. Let $A = \{0, 1, 2, \dots, 10\}$ and $R = \{(a,b) \in A \times A : 2|(a+b)\}$.

a) Show that R is an equivalence relation.
b) Compute the distinct equivalence classes of R and list the elements of every equivalence class.

18. Let $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ x^2 = y^2\}$.

a) Show that R is an equivalence relation.
b) Compute the distinct equivalence classes of R.

19. What are all possible equivalence relations on $A = \{1,2,3\}$?

20. For each of the following functions, determine if it is one-to-one, onto, or both. Prove your assertions.

a) $f: \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \textit{ is even} \\ \dfrac{x+1}{2} & \text{if } x \textit{ is odd} \end{cases}$$

b) $g: \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ defined by $f(n, m) = n - m$

c) $h: \mathbb{N} \times \{0,1\} \to \mathbb{Z}$ defined by $f(x, y) = \begin{cases} 2x & \text{if } y = 0 \\ 2x + 1 & \text{if } y = 1 \end{cases}$

21. Let A, B, C be sets, and suppose $f: A \to B$ and $g: B \to C$. Prove the following.
    a) If f and g are one-to-one, then $g \circ f$ is one-to-one.
    b) If f and g are onto, then $g \circ f$ is onto.
    c) If $g \circ f$ is one-to-one, then f is one-to-one. (Hint: Use contrapositive.)
    d) If $g \circ f$ is onto, then g is onto. (Hint: Use contrapositive.)

22. Give a combinatorial interpretation of the identity $2^n 2^m = 2^{n+m}$ for all $n, m \in \mathbb{N}$.

23. Consider the identity

$$k! \binom{n}{k} = n(n-1) \cdots (n-k+1)$$

    a) Prove this identity algebraically.
    b) Prove this identity combinatorially.

24. Consider the identity

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

    a) Prove this identity algebraically.
    b) Prove this identity combinatorially.

25. A long shelf on a math professor's library wall holds 28 books, all of them in the fields of algebra, calculus, or discrete math. Prove that there must be at least 10 books in one of these 3 areas.

26. A donut shop sells 9 varieties of donut. On a certain day 92 people buy at least one donut. What is the minimum number of donuts that must have been purchased for at least one of the varieties?

27. Let $A = \{a, b, c\}$ and $B = \{a, e, i, o, u\}$.
    a) How many relations are there from A to B?
    b) How many relations are there from B to A?
    c) How many functions are there from A to B?
    d) How many functions are there from B to A?
    e) How many one-to-one functions are there from A to B?
    f) How many one-to-one functions are there from B to A?

28. Prove that for every integer n,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

29. Suppose $a_1 = 3$ and $a_n = 2a_{n-1} - n + 2$ for $n \geq 2$. Prove that $a_n = 2^n + n$ for all positive integers n.

30. Prove that $\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n-1}{3} + \binom{n}{3} = \binom{n+1}{4}$ for every positive integer $n \geq 3$.

31. Suppose $b_1 = 3,\ b_2 = 9,$ and for $n \geq 3,\ b_n = 2b_{n-1} + 3b_{n-2}$. Prove that $b_n = 3^n$ for all positive integers n.

32. Prove that if $n \geq 18$, then you can make n cents out of just 4-cent and 7-cent stamps.
    Hint: Prove that the statement is true for $n = 18, 19, 20, 21$. Let $k \geq 21$, we assume that the statement is true for $18 \leq n \leq k$, prove that the statement is true for $n = k + 1$.