Calculus I (part 1): Limits and Continuity (by Evan Dummit, 2019, v. 2.50)

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1 Limits and Continuity

In this chapter, we introduce the fundamental idea of a limit, which captures the behavior of a function near a point of interest. Our primary interest in limits is to establish the definition of a continuous function, and to lay the technical groundwork for the definition of the derivative.

1.1 Limits (Informally)

- Informally, a limit is something that captures the "local" behavior of a function. Saying that the function $f(x)$ has the limit L as x approaches some value a means that as x gets "really close" to a, $f(x)$ gets "as close as we want" to L.
  - This idea may seem obvious or stupid at first (especially given that we have described it in rather vague and imprecise terms), but in fact it is more subtle than it might seem.
- Example: Consider the function $f(x) = \sqrt{x}$ as x approaches 9.
  - As x approaches 9, $f(x)$ should approach 3, based on our knowledge of the square root function.
  - We can compute $f(x)$ for some values of x close to 9:
    -

x	8	8.5	8.9	8.99	8.999
$f(x)$	2.828	2.915	2.983	2.998	2.9998

x	10	9.5	9.1	9.01	9.001
$f(x)$	3.162	3.082	3.016	3.0016	3.0001

  - And indeed, the values approach 3.
- Example: Consider the function $g(x) = \dfrac{x^2-1}{x-1}$, defined for $x \neq 1$, as x approaches 1.
  - Here is a short table of values:
    -

x	0	0.5	0.9	0.99	0.999	0.9999
$g(x)$	1	1.5	1.9	1.99	1.999	1.9999

x	2	1.5	1.1	1.01	1.001	1.0001
$f(x)$	3	2.5	2.1	2.01	2.001	2.0001

  - As x gets close to 1, it looks like $g(x)$ is approaching the value 2.
  - We can justify this with the following calculation: when $x \neq 1$ we can write $g(x) = \dfrac{(x-1)(x+1)}{(x-1)} = x+1$, since cancellation is allowed when the thing being cancelled is nonzero.
  - So as x gets close to 1, it makes sense that $g(x)$ gets close to 2.
- Example: Consider the function $h(x) = \dfrac{\sin(x)}{x}$, defined for $x \neq 0$, as x approaches 0.

- Here is a short table of values for positive x:
  -

x	1	0.5	0.1	0.05	0.01	0.001
$h(x)$	0.841	0.959	0.998	0.9996	0.99998	0.9999998

- Note that $h(x) = h(-x)$, so we will see the same behavior for negative x as for positive x.
- As x gets close to 0, it looks like $h(x)$ is approaching the value 1.
- However, unlike with $g(x)$ in the example above, there is no obvious algebraic manipulation we can perform that would explain why $h(x)$ is approaching 1 as x approaches 0.
- It certainly seems very plausible that as x takes even smaller values, $h(x)$ will continue getting closer and closer to 1, but at the moment we do not really have any way to justify this statement.

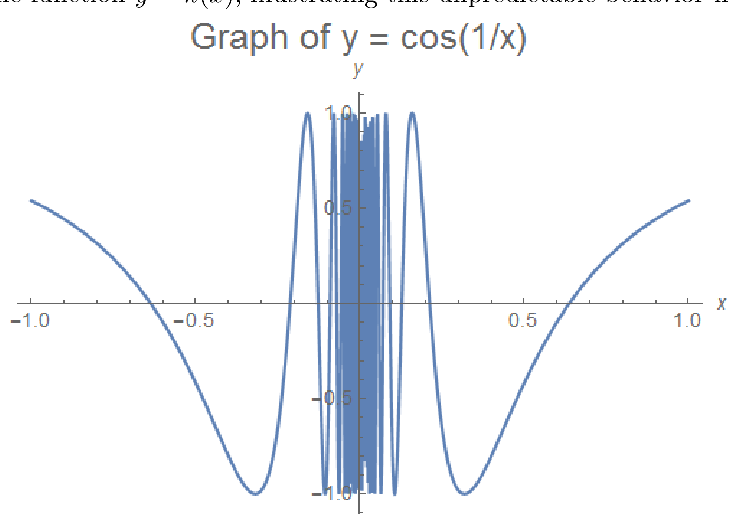
- Example: Consider $k(x) = \cos\left(\frac{1}{x}\right)$, defined for $x \neq 0$, as x approaches 0.
  - Here is a short table of values for positive x:
    -

x	1	0.5	0.1	0.05	0.01	0.001	0.0001	0.00001
$k(x)$	0.540	-0.416	-0.839	0.408	0.862	0.562	-0.9522	-0.9994

- Note that $k(x) = k(-x)$, so we will see the same behavior for negative x as for positive x.
- From these values it seems as though $k(x)$ is approaching the value -1 as x approaches 0.
- But it we compute $k(x)$ for even smaller x, we see that this table was misleading:
  -

x	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}
$k(x)$	0.937	-0.907	-0.363	0.838	0.873	0.370

- It appears that there is no pattern to the numbers as x approaches 0, even as x becomes very small.
- In hindsight, we should not have expected these numbers to have any kind of nice pattern: after all, as x approaches 0, $\frac{1}{x}$ becomes very large (positive or negative) and so the function $\cos\left(\frac{1}{x}\right)$ will oscillate wildly between the values -1 and $+1$ near $x = 0$.
- Here is a graph of the function $y = k(x)$, illustrating this unpredictable behavior near $x = 0$:



- As we can see from these examples, sometimes functions seem to have nice "limiting behavior" near particular values (even if those values are not actually in the domain of the function), and other times they do not.
  - Calculations like the ones we made in the examples can only take us so far.
  - For example, for $h(x) = \frac{\sin(x)}{x}$, how do we know that as we consider smaller and smaller values of x that $h(x)$ will continue getting closer to 1? After all, we made an erroneous guess for the very similar-looking function $k(x) = \cos\left(\frac{1}{x}\right)$: it looked initially like the function values were approaching -1 as x approached 0, but in fact they don't.

- To reiterate: how can we ever know that we've gotten "close enough" to conclude that the value tends to what our intuition says it does?
- We need "something else" (namely, a rigorous definition) that will allow us to see that our intuition about limiting values is correct.

1.2 Limits and the Limit Laws

- The formal definition of the limit allows us to back up our intuition with rigorous proof.
- Definition: A function $f(x)$ has the limit L as $x \to a$, written as $\lim_{x\to a} f(x) = L$, if, for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $0 < |x - a| < \delta$, we have that $|f(x) - L| < \epsilon$.
  - The symbols δ and ϵ are the lowercase Greek letters delta and epsilon (respectively). Their use in the definition of the limit is traditional. Also recall that the notation $|x|$ means the absolute value of x, and denotes the distance from x to zero.
  - Remark: This formal definition is very opaque. It takes practice and experience to become comfortable with what the definition means, and to see why it really does match the intuition of how a limit should behave.
  - One way to think of this definition is as follows: suppose you claim that the function $f(x)$ has a limit L, as x gets close to a. In order to prove to me that the function really does have that limit, I challenge you by handing you some value $\epsilon > 0$, and I want you to give me some open interval $(a - \delta, a + \delta)$ on the x-axis containing a, with the property that $f(x)$ is always within ϵ for x in that interval, except possibly at a.
  - If $f(x)$ really does stay close to the limit value L as x gets close to a, then, no matter what value of ϵ I picked, you should always be able to answer my challenge with an interval around a, because the values of $f(x)$ should stay near L when x is near a.
- We will not use the formal $\epsilon - \delta$ definition to calculate limits, as it is quite cumbersome even for very simple functions. Instead, we will state some properties that limits obey that can be used to reduce complicated limits to "simple" ones whose values we know.
  - Each of these properties can be justified using the formal definition, but the proofs are complicated. Furthermore, the details of the limit laws' proofs are not terribly important to understanding what the laws say and how they are used, which is why they are not given here.
- Theorem (Limit Laws): Let $f(x)$ and $g(x)$ be functions satisfying $\lim_{x\to a} f(x) = L_f$ and $\lim_{x\to a} g(x) = L_g$. Then the following properties hold:
  - Basic limits: If a and c are any constants, then $\lim_{x\to a} c = c$, and $\lim_{x\to a} x = a$.
  - The addition rule: $\lim_{x\to a} [f(x) + g(x)] = L_f + L_g$.
  - The subtraction rule: $\lim_{x\to a} [f(x) - g(x)] = L_f - L_g$.
  - The multiplication rule: $\lim_{x\to a} [f(x) \cdot g(x)] = L_f \cdot L_g$.
    * Note that the multiplication rule yields as a special case (when $g(x)$ is identically equal to a constant c) the constant-multiplication rule: $\lim_{x\to a} [c \cdot f(x)] = c \cdot L_f$, where c is any real number.
  - The division rule: $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L_f}{L_g}$, provided that L_g is not zero.
  - The exponentiation rule: $\lim_{x\to a} \left[f(x)^b\right] = (L_f)^b$, where b is any positive real number. (It also holds when b is negative or zero, provided L_f is positive, in order for both sides to be real numbers.)
  - The inequality rule: If $f(x) \le g(x)$ for all x, then $L_f \le L_g$.

- The squeeze rule (also called the sandwich rule): If $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ (meaning that both limits exist and are equal to L) then $\lim_{x \to a} g(x)$ exists and also equals L.

- Example: Evaluate $\lim_{t \to 1} \dfrac{t^3 - 3t + 1}{t - 3}$ formally using the limit laws.
  - First, we use the subtraction rule to see that $\lim_{t \to 1}(t - 3) = \lim_{t \to 1}(t) - \lim_{t \to 1}(3) = 1 - 3 = -2$, where we used the two "basic limits" at the end.
  - Next, we use the multiplication rule to see that $\lim_{t \to 1}(3t) = 3 \lim_{t \to 1}(t) = 3$, and the exponentiation rule to see that $\lim_{t \to 1}(t^3) = \lim_{t \to 1}(t)^3 = 1$.
  - Then we can use addition and subtraction to see that $\lim_{t \to 1}(t^3 - 3t + 1) = \lim_{t \to 1}(t^3) - \lim_{t \to 1}(3t) + \lim_{t \to 1}(1) = 1 - 3 + 1 = -1$.
  - Finally, we can use the division rule to see that $\lim_{t \to 1} \dfrac{t^3 - 3t + 1}{t - 3} = \dfrac{\lim_{t \to 1}(t^3 - 3t + 1)}{\lim_{t \to 1}(t - 3)} = \boxed{-\dfrac{1}{2}}$.
  - Note that this is the result we would have gotten if we had just "plugged in" $t = 1$ to the function.

- Example: Evaluate $\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right)$.
  - We cannot use the multiplication rule here, because, although $\lim_{x \to 0} x = 0$, $\lim_{x \to 0} \cos\left(\dfrac{1}{x}\right)$ does not exist, and the multiplication rule requires both limits to exist.
  - We use the squeeze theorem instead: because cosine is always between -1 and $+1$ and x^2 is always nonnegative, we have the inequalities $-x^2 \le x^2 \cos\left(\dfrac{1}{x}\right) \le x^2$.
  - Then because $\lim_{x \to 0}(-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$, which both also follow easily from the limit laws and the basic limits, the squeeze theorem dictates that $\boxed{\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right) = 0}$.

- In general, when evaluating limits using the limit laws, a reasonable procedure is first to try plugging in the limiting value to the function.
  - If plugging in yields a numerical value, then the limit can usually be evaluated by direct applications of the limit laws. (Using the limit laws in this way is essentially bookkeeping.)
  - If an indeterminate expression like $\dfrac{0}{0}$ is obtained, it is often necessary to perform some kind of algebraic manipulation first.

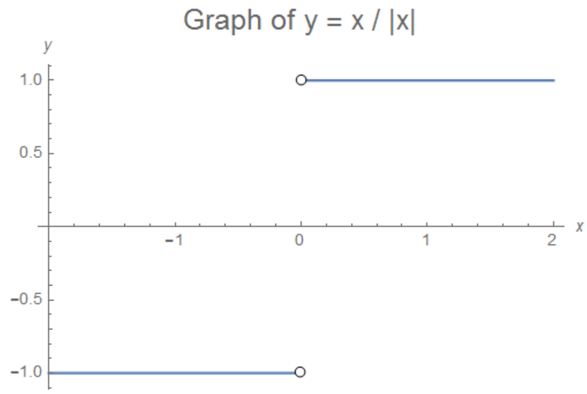
- Example: Evaluate $\lim_{t \to -2} \dfrac{t + 2}{t^2 - 4}$ formally using the limit laws.
  - If we try plugging in, we see that the numerator and denominator are both zero, suggesting that we should try to simplify the expression first.
  - Here, we can factor the denominator: $t^2 - 4 = (t - 2)(t + 2)$.
  - Then we can write $\lim_{t \to -2} \dfrac{t + 2}{t^2 - 4} = \lim_{t \to -2} \dfrac{t + 2}{(t - 2)(t + 2)} = \lim_{t \to -2} \dfrac{1}{t - 2}$, where we cancelled the common factor.
  - Finally, the subtraction and division laws give us $\lim_{t \to -2} \dfrac{1}{t - 2} = \boxed{-\dfrac{1}{4}}$.
  - Remark: The use of the limit allows us to say something about the function $f(t) = \dfrac{t + 2}{t^2 - 4}$ near $t = -2$, even though the function (as written) is not defined there.

- Example: Evaluate $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.
  - If we try plugging in, we see that the numerator and denominator are both zero, suggesting that we should try to simplify the expression first.
  - Here, the denominator is a difference involving square roots. A useful technique in this situation is to "multiply by the conjugate":
  $$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{(\sqrt{x})^2-3^2}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}.$$
  - Now we can cancel the common factor $x-9$ and are left with the simple limit $\lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \boxed{\frac{1}{6}}$.

1.3 One-Sided Limits

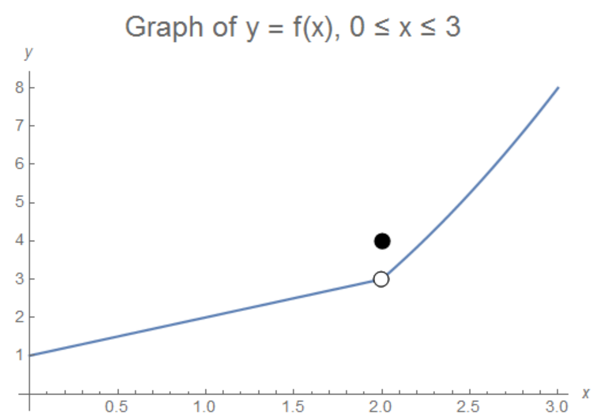
- Sometimes, we are only interested in the behavior of a limit as the parameter approaches from one side – for example, we might have a physical system and only possess data over some time interval, and would like to say things about how the system behaves at the ends of the time interval.
- This motivates us to define "one-sided limits". The definitions are the following (although, as before, we will not use the definition to evaluate limits):
  - We say that a function $f(x)$ has the limit L as $x \to a$ from below, written as $\lim_{x\to a-} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $-\delta < x-a < 0$, we have that $|f(x)-L| < \epsilon$.
  - We say that a function $f(x)$ has the limit L as $x \to a$ from above, written as $\lim_{x\to a+} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $0 < x-a < \delta$, we have that $|f(x)-L| < \epsilon$.
  - The only change in the definition is the restriction on the x-axis: a two-sided limit has $0 < |x-a| < \delta$, while the limit from below has $0 < a-x < \delta$ and the limit from above has $0 < x-a < \delta$.
  - Remark: "Normal" limits are sometimes called two-sided limits, to distinguish them from the two possible one-sided limits. The term "limit", with no qualifier, refers to a two-sided limit.
- All of the limit laws also hold for one-sided limits, with all of the limits being changed to the appropriate one-sided limit.
- Example: Find $\lim_{x\to 0+} \sqrt{x}$.
  - We know that $\lim_{x\to 0+} x = 0$, and so by the exponentiation rule we see $\lim_{x\to 0+} \sqrt{x} = \sqrt{\lim_{x\to 0+} x} = \boxed{0}$.
  - Depending on one's preference regarding complex numbers, one might say that the two-sided $\lim_{x\to 0} \sqrt{x}$ does not exist (because $\sqrt{x}$ is not a real number for negative x). Here, the one-sided limit avoids these difficulties.
- Proposition (One- and Two-Sided Limits): A two-sided limit exists precisely when both one-sided limits exist and have the same value. When the two one-sided limits are equal, the two-sided limit shares the same value.
  - This result is more or less a restatement of the definitions of the one-sided limits.
  - In particular: a two-sided limit won't exist if either of the one-sided limits doesn't exist, or if the two one-sided limits have different values.
  - Some functions have different behaviors to the left and to the right of a given point, which will be detected by the one-sided limits.
- Example: Find the one-sided limits $\lim_{x\to 0+} \frac{x}{|x|}$ and $\lim_{x\to 0-} \frac{x}{|x|}$. Does the limit $\lim_{x\to 0} \frac{x}{|x|}$ exist?

- We can see that $\dfrac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$, so the graph of this function jumps from -1 to $+1$ at $x = 0$:



- By our results on limits of constants, we see that $\lim_{x \to 0+} \dfrac{x}{|x|} = \boxed{1}$, while $\lim_{x \to 0-} \dfrac{x}{|x|} = \boxed{-1}$.
- The (two-sided) limit of this function at 0 therefore $\boxed{\text{does not exist}}$, since the one-sided limits have different values.

- Example: Let $f(x) = \begin{cases} x^2 - 1 & \text{if } x > 2 \\ 4 & \text{if } x = 2 \\ x + 1 & \text{if } x < 2 \end{cases}$. Find $\lim_{x \to 2+} f(x)$, $\lim_{x \to 2-} f(x)$, and $\lim_{x \to 2} f(x)$.
  - Here is a graph of this function:



  - When $x > 2$, $f(x) = x^2 - 1$, so $\lim_{x \to 2+} (x^2 - 1) = \boxed{3}$ using the limit laws.
  - Similarly, when $x < 2$, $f(x) = x + 1$, so $\lim_{x \to 2-} (x + 1) = \boxed{3}$ using the limit laws.
  - Since the two one-sided limits are equal, the two-sided limit exists and has the same value: $\lim_{x \to 2} f(x) = \boxed{3}$.
  - Notice that although $f(2) = 4$, the limit $\lim_{x \to 2} f(x)$ is equal to 3.
  - This odd behavior is reflected in the graph of $y = f(x)$, and explains why there is a "hole": the curves representing $y = x^2 - 1$ for $x > 2$ and $y = x + 1$ for $x < 2$ both approach the point $(2, 3)$ as $x \to 2$. However, because $f(2)$ is actually equal to 4, the point $(2, 3)$ is missing from the graph of $y = f(x)$, as it has been replaced with the point $(2, 4)$.

1.4 Continuity

- We will now discuss a very important property of functions that captures a notion of "niceness".
- Definition: A function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$. A function $f(x)$ is called everywhere continuous (or often, just "continuous") if it is continuous at $x = a$ for all real numbers a in its domain.

- Continuous functions are (generally speaking) much nicer than arbitrary functions. Continuous functions do not "jump" or have "missing points", nor do they blow up to ∞.
- A common interpretation of a continuous function is: its graph can be drawn without taking the pencil off of the paper[1].
- Most of the functions we will encounter (polynomials, trigonometric functions, exponentials and logarithms) will be continuous everywhere they are defined, except possibly at a small number of points.

- Proposition: Every polynomial is everywhere continuous.
  - Proof: We want to show that $\lim_{x \to a} p(x) = p(a)$, where $p(x)$ is any polynomial in x, and a is any real number.
  - By the exponentiation rule, we know that $\lim_{x \to a} x^n = \left(\lim_{x \to a} x\right)^n = a^n$, for any positive integer n, since we already know that $\lim_{x \to a} x = a$.
  - By the multiplication rule, we know that $\lim_{x \to a} (c_n x^n) = \left[\lim_{x \to a} c_n\right] \cdot \left[\lim_{x \to a} x^n\right] = c_n a^n$, for any constant c_n, since we know that $\lim_{x \to a} c_n = c_n$.
  - Finally, because every polynomial is just a sum of terms of the form $c_n x^n$ for some coefficients c_n, we can use the addition rule (repeatedly) to conclude that $\lim_{x \to a} p(x) = p(a)$, where $p(x)$ is an arbitrary polynomial.

- Proposition: The functions e^x, $\sin(x)$, and $\cos(x)$ are continuous for all x.
  - The proof of this result requires using the technical definitions of these functions, either as limits or as infinite sums.

- Theorem: If $f(x)$ and $g(x)$ are continuous functions, then $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, and $f(g(x))$ are continuous functions. If $g(x) \neq 0$ then $\dfrac{f(x)}{g(x)}$ is also continuous, and if $g(x) > 0$ then $[g(x)]^a$ is continuous, for any real number a. If $f(x)$ is one-to-one then $f^{-1}(x)$ is also continuous.
  - The proofs for the sum, difference, product, quotient, and exponentiation follow from the respective limit laws.
  - The proof for the composition uses the definition of continuity twice.
  - The proof for the inverse function uses the intermediate value theorem (see below).
  - Note that a quotient of functions is never continuous anywhere that the denominator is zero (since, although the limit may exist, the quotient itself is not defined and therefore cannot equal the limit).

- Proposition: Any quotient of polynomials $\dfrac{p(x)}{q(x)}$ is continuous everywhere it is defined, as are the trigonometric functions $\tan(x)$, $\sec(x)$, $\csc(x)$, $\cot(x)$, and the logarithm $\ln(x)$.
  - These results follow from the properties of continuous functions and the results that polynomials, $\sin(x)$, $\cos(x)$, and e^x are continuous.
  - Example: The function $f(x) = \dfrac{1}{x}$ is continuous at all $x \neq 0$, and is discontinuous at $x = 0$.

- Theorem (Continuity and Limits): If $g(x)$ is a continuous function, and $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} g(f(x)) = g(L)$.
  - In other words, we can move continuous functions through limits without having to worry about changing the value.

[1]Technically, this description is slightly stronger than continuity: it is describing a continuous function of "bounded variation". The problem is that the graph could have an infinite length even on a finite interval (and such a graph cannot be physically drawn using a real pencil). One example is $f(x) = \begin{cases} x\sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, which is continuous at $x = 0$ by the squeeze theorem, but the length of the graph of $y = f(x)$ on any open interval containing $x = 0$ is infinite.

- The proof is very similar to the proofs of the limit laws.

- Using properties of continuous functions can allow us to evaluate many complicated limits with ease:

- Example: Find $\lim_{x\to\pi} \tan^{-1}\left(\dfrac{e^{\sin(x)}}{\sqrt{2-\cos(x)}}\right)$.

  - First, because $e^{\sin(x)}$ and $\sqrt{2-\cos(x)}$ are both continuous functions, and the latter function is never zero (since $\cos(x)$ is between -1 and 1), the quotient $\dfrac{e^{\sin(x)}}{\sqrt{2-\cos(x)}}$ is also continuous.
  - Thus, $\lim_{x\to\pi} \dfrac{e^{\sin(x)}}{\sqrt{2-\cos(x)}} = \dfrac{e^{\sin(\pi)}}{\sqrt{2-\cos(\pi)}} = \dfrac{e^0}{\sqrt{2-(-1)}} = \dfrac{1}{\sqrt{3}}$.
  - Then, since arctangent is continuous, we can move it through the limit to write

$$\lim_{x\to\pi} \tan^{-1}\left(\frac{e^{\sin(x)}}{\sqrt{2-\cos(x)}}\right) = \tan^{-1}\left(\lim_{x\to\pi}\frac{e^{\sin(x)}}{\sqrt{2-\cos(x)}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}.$$

- A fundamental property is that a continuous function has no "jumps". The formal statement is as follows:

- Intermediate Value Theorem: If the function $f(x)$ is continuous on the interval $[a,b]$ then for any y between $f(a)$ and $f(b)$, there exists a value c in (a,b) such that $y = f(c)$. Equivalently, f attains all values between $f(a)$ and $f(b)$ as x goes from a to b.

  - This theorem is essentially equivalent to the formal definition of the real numbers, and its proof requires using a technical result known as the least upper bound axiom.
  - Using the Intermediate Value Theorem, we can prove that continuous functions must take on particular values, even if we cannot explicitly say exactly where.

- Example: Show that there is a real number c such that $c = \cos(c)$.

  - If we let $f(x) = x - \cos(x)$, then we want to show that $f(x)$ attains the value zero somewhere.
  - We can compute $f(0) = -1$ and $f(\pi) = \pi + 1$.
  - Since $f(x)$ is continuous, and 0 lies between $f(0)$ and $f(1)$, the Intermediate Value Theorem dictates that there is some c in $(0,\pi)$ such that $f(c) = 0$.
  - For this value of c, we have $0 = f(c) = c - \cos(c)$, so that $c = \cos(c)$ as desired.

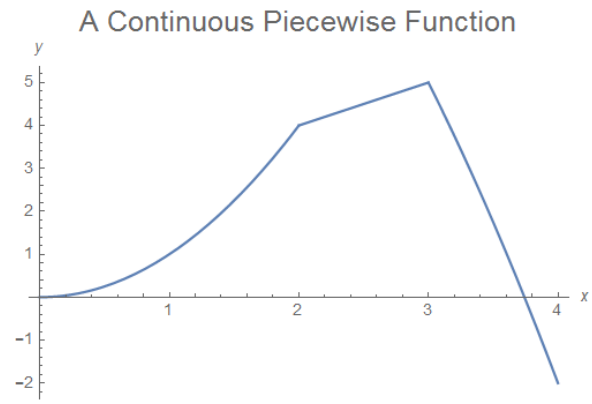
- Example: Show that the function $p(x) = x^7 - 8x + 6$ has at least three real zeroes.

  - We will use the Intermediate Value Theorem on three separate intervals to show that there must exist a zero in each interval.
  - If we make a small table of values, we will see that $p(-2) = -106$, $p(-1) = 13$, $p(0) = 6$, $p(1) = -1$, and $p(2) = 118$.
  - Notice that $p(x)$ is continuous and changes sign in each of the intervals $[-2,-1]$, $[0,1]$, and $[1,2]$.
  - Thus, applying the Intermediate Value Theorem on each interval shows that $p(x)$ has at least one real zero in each of the intervals $(-2,-1)$, $(0,1)$, and $(1,2)$, so it must have at least 3 zeroes in total.

- With all of these results, essentially the only remaining simple functions for which continuity is still an interesting question are "piecewise-defined" functions. For such functions, to determine continuity requires comparing one-sided limits to the value of the function at the points where the definition of the function changes.

- Example: Determine the values of a and b that make $f(x) = \begin{cases} x^2 & \text{for } x < 2 \\ x + a & \text{for } 2 \le x \le 3 \\ b - x^2 & \text{for } x > 3 \end{cases}$ continuous for all x.

- On the intervals $(-\infty, 2)$, $(2, 3)$, and $(3, \infty)$, f is defined as a single polynomial, and is therefore continuous. The only possible issues are at $x = 2$ and $x = 3$, where the definition of f changes from one polynomial to another.
- As $x \to 2$ from below, $f(x) \to 4$. As $x \to 2$ from above, $f(x) \to 2 + a$, and $f(2) = 2 + a$. We need all three values to be equal, so $4 = a + 2$ so that $a = 2$.
- Now as $x \to 3$ from below, $f(x) \to 3 + a = 5$. As $x \to 3$ from above, $f(x) \to b - 9$, and $f(3) = 3 + a = 5$. We need all three values to be equal, so $5 = b - 9$ so that $b = 14$.
- Therefore, $f(x)$ will be continuous when $\boxed{a = 2,\ b = 14}$.
- Here is a graph of $y = f(x)$, for $a = 2$ and $b = 14$:



1.5 Limits at Infinity, Infinite Limits

- Another type of behavior we would like to study is how a function $f(x)$ behaves when the value of x becomes very large (either large positive or large negative).
- The precise definitions (which, again, we will not use for evaluating limits) are as follows:
  - We say that a function $f(x)$ <u>has the limit L as $x \to +\infty$</u>, written as $\lim_{x\to+\infty} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists an $A > 0$ (depending on ϵ) with the property that for all $x > A$, we have that $|f(x) - L| < \epsilon$.
  - We say that a function $f(x)$ <u>has the limit L as $x \to -\infty$</u>, written as $\lim_{x\to-\infty} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists an $A > 0$ (depending on ϵ) with the property that for all $x < -A$, we have that $|f(x) - L| < \epsilon$.
  - <u>Notation</u>: The symbols ∞ and $+\infty$ mean the same thing ("positive infinity"); the $+\infty$ is often used for contrast with the $-\infty$ symbol ("negative infinity").
- All of the limit laws also hold for limits at infinity, with all of the limits being changed to the appropriate infinite limit.
- Note that some functions, such as $f(x) = x$, do not tend to a finite value as x becomes very large, but instead "grow to infinity". Other functions, such as $g(x) = \frac{1}{x^2}$ and $h(x) = \ln(x)$, become very large even for finite values of x. So we will define "infinite limits" to cover those cases:
  - We say that a function $f(x)$ <u>diverges to $+\infty$ as $x \to c$</u>, written as $\lim_{x\to c} f(x) = +\infty$, if the following statement is true: for any $B > 0$ (no matter how large) there exists a $\delta > 0$ (depending on B) with the property that for all $0 < |x - c| < \delta$, we have $f(x) > B$.
    - The idea of diverging to $-\infty$ is analogous, except instead $f(x) < -B$.
    - We can also formulate one-sided limits here, with $x \to c$ from above or from below.

- We say that a function $f(x)$ diverges to $+\infty$ as $x \to \infty$, written as $\lim_{x\to\infty} f(x) = +\infty$, if the following statement is true: for any $B > 0$ (no matter how large) there exists an $A > 0$ (depending on B) with the property that for all $x > A$, we have $f(x) > B$.
  - The statements for diverging to $-\infty$, or the statements as $x \to -\infty$, are similar.

- There are some additional limit laws that allow us to work with infinite limits (where a can be finite or infinite):
  - Basic Limits: $\lim_{x\to+\infty} c = c$ for any constant c, $\lim_{x\to+\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} x = \infty$, and $\lim_{x\to 0+} \ln(x) = -\infty$.
  - Negation: If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} [-f(x)] = -\infty$, and vice versa.
  - Multiplication: If $\lim_{x\to a} f(x)$ is a finite positive number or ∞ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x)g(x)] = \infty$.
  - Addition: If $\lim_{x\to a} f(x)$ is a finite number or ∞ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x) + g(x)] = \infty$.
  - Division: If $\lim_{x\to a} f(x)$ is a finite number and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 0$.
  - Exponentiation (∞^L): If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = L$ where $L > 0$, or $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)^{g(x)}$ is ∞.
  - Exponentiation (L^∞): If $\lim_{x\to a} f(x) = L$ (where $L \geq 0$) and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)^{g(x)}$ is 0 if $0 \leq L < 1$, and is ∞ if $L > 1$.

- Using these limit laws for ∞, we can attach a concrete meaning to certain arithmetic operations involving ∞ that allows us to treat ∞ and $-\infty$ as if they were "almost" real numbers (subject to some restrictions):
  - The "negation" statement says that $-1 \cdot \infty = -\infty$ (i.e., we can move minus signs to and from ∞ in the way we would expect).
  - The "multiplication" statement says $\infty \cdot \infty$ and $c \cdot \infty$ are both ∞ for any positive constant c.
  - The "addition" statement says that $c + \infty$ and $\infty + \infty$ are both ∞ for any finite c.
  - The "division" statement says that $\frac{c}{\infty} = 0$ for any finite c.
  - The "exponentiation" statements say that a^∞ is 0 if $0 < a < 1$ and is ∞ if $a > 1$, and also that $\infty^a = \infty$ for $a > 0$. (Both statements include the fact that $\infty^\infty = \infty$.)
  - By combining some of the above statements we can see what happens with $-\infty$ as well: for example, $c - \infty = -\infty$ and $(-\infty) \cdot (-\infty) = \infty$, and $a^{-\infty}$ is ∞ if $a < 1$ and is 0 if $a > 1$.
  - The other possible arithmetic operations (namely, $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, 1^∞, and ∞^0) are "indeterminate forms", meaning that their values depend on the functions whose limits are involved. (They may be finite, infinite, or not exist at all, depending on the situation.)

- Many infinite limits can be evaluated using direct appeals to the limit laws and the basic limits.

- Example: Find $\lim_{x\to\infty} x^n$ and $\lim_{x\to-\infty} x^n$, for n a positive integer.
  - By applying the exponentiation rule (∞^L) for limits with $f(x) = x$ and $g(x) = n$, we see that $\lim_{x\to\infty} x^n = \boxed{\infty}$.
  - For the other limit, observe that $x^n = (-1)^n(-x)^n$. By applying the exponentiation rule (∞^L) for limits with $f(x) = -x$ and $g(x) = n$ (note that $f(x)$ is positive when x is large and negative), we see that $\lim_{x\to-\infty} (-x)^n = \infty$.
  - Thus, $\lim_{x\to-\infty} x^n = (-1)^n \lim_{x\to-\infty} (-x)^n = \boxed{\begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}}$.

- Example: Find $\lim_{x\to a+} \frac{1}{(x-a)^n}$, $\lim_{x\to a-} \frac{1}{(x-a)^n}$, and $\lim_{x\to a} \frac{1}{(x-a)^n}$, for n a positive integer.
  - By applying the exponentiation rule (L^∞) for limits with $f(x) = \frac{1}{x-a}$ (which is positive when $x > a$) and $g(x) = n$, we see that $\lim_{x\to a+} \frac{1}{(x-a)^n} = \boxed{\infty}$.
  - For the other limit, observe that $(x-a)^n = (-1)^n(a-x)^n$. Applying the exponentiation rule (L^∞) for limits with $f(x) = \frac{1}{a-x}$ (which is positive when $x < a$) and $g(x) = n$, we see that $\lim_{x\to a-} \frac{1}{(a-x)^n} = \infty$.
  - Then $\lim_{x\to a-} \frac{1}{(x-a)^n} = (-1)^n\infty = \boxed{\begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}}$.
  - Finally, by comparing the two one-sided limits, we see $\lim_{x\to a-} \frac{1}{(x-a)^n} = \begin{cases} \infty & \text{if } n \text{ is even} \\ \text{DNE} & \text{if } n \text{ is odd} \end{cases}$.
- Example: Find $\lim_{x\to 0} \frac{1}{x^2}$.
  - This example is merely a special case of the previous one, with $a = 0$ and $n = 2$.
  - From the analysis above, we see that the limit exists and equals $\boxed{\infty}$.
- Example: Find $\lim_{x\to\infty} e^x$ and $\lim_{x\to-\infty} e^x$.
  - By applying the exponentiation rule for infinite limits we see $\lim_{x\to\infty} e^x = \boxed{\infty}$.
  - For the other limit, we could either use the exponentiation rule again, or use the division rule to see that $\lim_{x\to-\infty} e^x = \lim_{y\to\infty} e^{-y} = \lim_{y\to\infty} \frac{1}{e^y} = \boxed{0}$ (where we set $y = -x$ in the first step).
- Example: Find $\lim_{x\to 2} \frac{x^2-5}{(x-2)^6}$.
  - Note that the limit of the numerator as $x \to 2$ is equal to -1, while the denominator goes to zero.
  - From our earlier analysis, we know that $\lim_{x\to 2} \frac{1}{(x-2)^6} = \infty$.
  - Thus, by the multiplication rule for infinite limits, we see that $\lim_{x\to 2} \frac{x^2-5}{(x-2)^6} = (-1)\cdot(\infty) = \boxed{-\infty}$.
- Other limits require some algebraic manipulation or simplification to evaluate.
  - In general, a good heuristic is that the behaviors of infinite limits tend to be determined by the largest terms.
  - For limits involving quotients of polynomials, or in general functions that can be compared to powers of x, a useful idea is to factor out the highest power of x that appears in the numerator and the denominator.
- Example: Find $\lim_{x\to\infty} (x^3 - 2x^2 + 4)$ and $\lim_{x\to-\infty} (x^3 - 2x^2 + 4)$.
  - We should expect that the behavior of the limit will be determined by the largest quantity, which in this case is x^3.
  - If we factor out the largest power of x, we can write $x^3 - 2x^2 = x^3 \cdot (1 - 2/x + 4/x^3)$.
  - Taking the limit as $x \to \infty$, we see that $x^3 \to \infty$ while $(1 - 2/x + 4/x^3) \to 1$. Thus, by the infinite limit laws for products, we see that $\lim_{x\to\infty} (x^3 - 2x^2 + 4) = \boxed{\infty}$.
  - Similarly, if we take the limit as $x \to -\infty$, we see that $x^3 \to -\infty$ while $(1 - 2/x + 4/x^3) \to 1$. Thus, by the infinite limit laws for products, we see that $\lim_{x\to-\infty} (x^3 - 2x^2 + 4) = \boxed{-\infty}$.

- Example: Evaluate $\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2-x-71}$.
  - Here, x^2 is the highest power that appears in both the numerator and denominator.
  - Factoring it out yields $\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2-x-71} = \lim_{x\to\infty} \frac{x^2\cdot\left[3+2/x-1/x^2\right]}{x^2\cdot\left[1-1/x-71/x^2\right]}$.
  - Now we can use the limit laws repeatedly to break down the evaluation into easy pieces, to see that $\lim_{x\to\infty}\left[3+2/x-1/x^2\right]=3$ and $\lim_{x\to\infty}\left[1-1/x-71/x^2\right]=1$, and so we obtain $\boxed{\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2-x-71}=3}$.

- Example: Evaluate $\lim_{x\to\infty} \frac{3x^3+2x-1}{x^2-x-71}$.
  - Like above, we pull out the highest power of x that appears in the numerator and the denominator.
  - This gives $\lim_{x\to\infty} \frac{3x^3+2x-1}{x^2-x-71} = \lim_{x\to\infty} \frac{x^3\cdot\left[3+2/x^2-1/x^3\right]}{x^2\cdot\left[1-1/x-71/x^2\right]} = \lim_{x\to\infty} x\cdot\frac{3+2/x^2-1/x^3}{1-1/x-71/x^2}$.
  - We have $\lim_{x\to\infty} x=\infty$ and $\lim_{x\to\infty}\frac{3+2/x^2-1/x^3}{1-1/x-71/x^2}=3$ by the same kind of argument as before, so by the multiplication rule for infinite limits we see that the original limit was $\boxed{\infty}$.

- Example: Find $\lim_{x\to\infty}\frac{\sqrt{x^2+4}}{x-2}$ and $\lim_{x\to-\infty}\frac{\sqrt{x^2+4}}{x-2}$.
  - In each case, observe that the numerator and denominator both go to $\pm\infty$. To evaluate the limit, we pull out the largest power of x from each.
  - For the first limit, $\lim_{x\to\infty}\frac{\sqrt{x^2+4}}{x-2} = \lim_{x\to\infty}\frac{\sqrt{x^2(1+4/x^2)}}{x(1-2/x)} = \lim_{x\to\infty}\frac{\sqrt{x^2}\cdot\sqrt{1+4/x^2}}{x(1-2/x)}$.
  - Since x is large and positive, $\sqrt{x^2}=x$, so $\lim_{x\to\infty}\frac{\sqrt{x^2}\cdot\sqrt{1+4/x^2}}{x(1-2/x)} = \lim_{x\to\infty}\frac{\sqrt{1+4/x^2}}{1-2/x} = \frac{1}{1} = \boxed{1}$.
  - For the second limit, the same calculation yields $\lim_{x\to-\infty}\frac{\sqrt{x^2+4}}{x-2} = \lim_{x\to-\infty}\frac{\sqrt{x^2}\cdot\sqrt{1+4/x^2}}{x(1-2/x)}$.
  - Since x is large and negative, $\sqrt{x^2}=-x$, so $\lim_{x\to\infty}\frac{\sqrt{x^2}\cdot\sqrt{1+4/x^2}}{x(1-2/x)} = \lim_{x\to\infty}\frac{-\sqrt{1+4/x^2}}{1-2/x} = \frac{-1}{1} = \boxed{-1}$.

- Example: Find $\lim_{x\to\infty}\left[\sqrt{x^2+6x}-x\right]$.
  - Here, there is a difference involving a square root, so we use the idea of "multiplying by the conjugate":

$$\lim_{x\to\infty}\left[\sqrt{x^2+6x}-x\right] = \lim_{x\to\infty}\frac{\sqrt{x^2+6x}-x}{1}\cdot\frac{\sqrt{x^2+6x}+x}{\sqrt{x^2+6x}+x} = \lim_{x\to\infty}\frac{(x^2+6x)-x^2}{\sqrt{x^2+6x}+x} = \lim_{x\to\infty}\frac{6x}{\sqrt{x^2+6x}+x}.$$

  - Now we can pull out the largest power of x from the numerator and denominator:

$$\lim_{x\to\infty}\frac{6x}{\sqrt{x^2+6x}+x} = \lim_{x\to\infty}\frac{x\cdot 6}{x\sqrt{1+6/x}+x} = \lim_{x\to\infty}\frac{6}{\sqrt{1+6/x}+1} = \frac{6}{1+1} = \boxed{3}.$$

Well, you're at the end of my handout. Hope it was helpful.