# Contents



# 0 Review of Basic Concepts

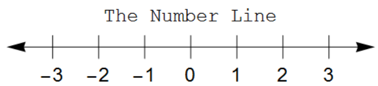
In this chapter, we review a variety of basic mathematical concepts that will be needed to discuss calculus. All of these topics are treated very superficially, as our goal is only to provide a brisk review of the necessary material.

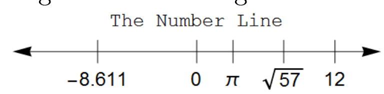
## 0.1 Numbers, Sets, and Intervals

- Definition: The positive integers (1, 2, 3, ...) are obtained by adding 1 to itself some number of times; the other integers are 0 and the negatives of the positive integers ($-1,-2,-3$, ...). Integers can be added (+), subtracted ($-$), and multiplied ($\cdot$), but not always divided (/). Integers are also called whole numbers, and the positive integers are also called the natural numbers.
  - Examples of integers: 3, 0, $-666$, 1337, $10^{10^{10}}$.
- Definition: The rational numbers are numbers of the form $a/b$ where $a$ and $b$ are integers and $b$ is not zero. Rational numbers can be added, subtracted, multiplied, and always divided (except by 0).
  - Examples of rational numbers: $\frac{1}{2}$, $-\frac{225}{1037}$, 11, 0.
  - The basic operations are $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ and $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.
- Definition: The real numbers are harder to define simply, but roughly speaking, the real numbers are obtained by filling in the "gaps" between the rational numbers. For example, there is no positive rational number whose square is exactly equal to 2; this "missing" number is the real number $\sqrt{2}$. Another way to think of real numbers is as (finite or infinite) decimal sequences[1]. Real numbers can be added, subtracted, multiplied, and always divided (except by 0).

[1] Some real numbers have two decimal sequences: 1.000... = 0.999... are two different ways of writing the positive integer 1. A similar ambiguity occurs with any other decimal number ending in an infinite string of 9s, but these are the only real numbers with two decimal representations. We will not dwell further on this technical point at the moment.

- Examples of real numbers: $\pi$, $\sqrt{2}$, $\dfrac{\pi+\sqrt{2}}{5}$, 11, 0, 0.12131415161718192021....
- Rational numbers are those real numbers that have (eventually) repeating or terminating decimal sequences. For example, $\dfrac{4}{125} = 0.032$, and $\dfrac{3}{11} = 0.27272727... = 0.\overline{27}$, where the bar indicates that portion repeats forever.
- A common way to visualize the real numbers is as the "number line", with negative numbers of increasing magnitude on the left and positive numbers of increasing magnitude on the right:




- Definition: The complex numbers are of the form $a + ib$ where $a$ and $b$ are real numbers and $i$ is the so-called "imaginary unit" satisfying $i^2 = -1$.
  - Examples of complex numbers: $3 + 4i$, $\sqrt{-2}$, $\pi$, 11, 0, $(1-i)^{37}$.
- Definition: A set is a well-defined collection of distinct elements.
  - The elements of a set can be essentially anything: integers, real numbers, other sets, people. We are generally interested in sets of real numbers.
  - Sets are generally denoted by capital or script letters, and when listing the elements of a set, curly brackets $\{\cdot\}$ are used.
  - Sets do not have to have any elements: the empty set $\emptyset = \{\ \}$ is the set with no elements at all.
  - Two sets are the same precisely if all of their elements are the same. The elements in a set are also not ordered, and no element can appear in a set more than once: thus the sets $\{1,4\}$ and $\{4,1\}$ are the same.
- There are two primary ways to describe a set.
  - One way is to list all the elements: for example, $A = \{1,2,4,5\}$ is the set containing the four numbers 1, 2, 4, and 5.
  - The other way to define a set is to describe properties of its elements[2]: for example, the set $S$ of one-letter words in the English alphabet has two elements: $S = \{a, I\}$.
- We often employ "set-builder" notation for sets: for example, the set $S$ of real numbers between 0 and 5 is denoted $S = \{x : x \text{ is a real number and } 0 \le x \le 5\}$.
  - Some authors use a vertical pipe $|$ instead of a colon : but this distinction is irrelevant.
- Sets of common types of numbers come up very often, and are given special symbols:
  - The set $\{1,2,3,...\}$ of all natural numbers is denoted $\mathbb{N}$ ("naturals").
  - The set $\{..., -2, -1, 0, 1, 2, ...\}$ of all integers is denoted $\mathbb{Z}$ (*Zahlen*, German for "numbers").
  - The set of all rational numbers is denoted $\mathbb{Q}$ ("quotients").
  - The set of all real numbers is denoted $\mathbb{R}$ ("reals").
  - The set of all complex numbers is denoted $\mathbb{C}$ ("complex").
- There is additional ("interval") notation for subsets of the real numbers. For real numbers $a, b$ with $a < b$:
  - The "closed interval" $[a,b]$ denotes the set of real numbers $x$ satisfying $a \le x \le b$.
  - The "open interval" $(a,b)$ denotes the set of real numbers $x$ satisfying $a < x < b$.
  - The half-open interval $(a,b]$ denotes the set of real numbers satisfying $a < x \le b$.
  - The half-open interval $[a,b)$ denotes the set of real numbers satisfying $a \le x < b$.

[2] It is possible to run into trouble by trying to define sets in this "naive" way of specifying qualities of their elements. In general, one must be more careful when defining arbitrary sets, although we will not worry about this.

- The notation $[a, \infty)$ denotes the set of real numbers $x$ satisfying $a \leq x$.
- The notation $(-\infty, b]$ denotes the set of real numbers $x$ satisfying $x \leq b$.
  - There are also open versions (with a round bracket) which do not include the finite endpoint.
  - Note that $\infty$ and $-\infty$ are not real numbers: they are just suggestive pieces of notation.
- The notation $(-\infty, \infty)$ denotes the set of all real numbers.

- Notation: If $S$ is a set, $x \in S$ means "$x$ is an element of $S$", and the notation $x \notin S$ means "$x$ is not an element of $S$".
  - Example: For $S = \{1, 2, 5\}$ we have $1 \in S$ and $5 \in S$ but $3 \notin S$ and $\pi \notin S$.

- Definition: If $A$ and $B$ are two sets with the property that every element of $A$ is also an element of $B$, we say $A$ is a subset of $B$ (or that $A$ is contained in $B$) and write $A \subseteq B$.
  - Example: If $A = \{1, 2, 3\}$, $B = \{1, 4, 5\}$, and $C = \{1, 2, 3, 4, 5\}$, then $A \subseteq C$ and $B \subseteq C$ but neither $A$ nor $B$ is a subset of the other.
  - Note: Subset notation is not universally agreed-upon: the notation $A \subset B$ is also commonly used to say that $A$ is a subset of $B$.
  - The difference is not terribly relevant except for when $A$ can be equal to $B$: some authors allow $A \subset B$ to include the possibility that $A$ could be equal to $B$, while others insist that $A \subset B$ means that $A$ is a subset of $B$ which cannot be all of $B$.

- Definition: If $A$ and $B$ are two sets, then the intersection $A \cap B$ is the set of all elements contained in both $A$ and $B$. The union $A \cup B$ is the set of all elements contained in either $A$ or $B$ (or both).
  - Example: If $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$, then $A \cap B = \{1\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$.

## 0.2 Functions

- Definition: A function is a relation between a set of inputs (called the domain of the function) and a set of outputs (called the range of the function): to each element of the domain, the function associates a single value in the range.
  - Example: Consider $f(x) = x^3$, with domain and range both the set of real numbers. This function $f$ sends each real number $x$ to its cube $x^3$: thus $f(2) = 8$, $f(0) = 0$, and $f(-1) = -1$.
  - We will usually work with functions whose domain and range are (subsets of) the real numbers. But functions can be defined with any arbitrary domain and range.
  - In general, unless specified, the domain of a function is the largest possible set of real numbers for which the definition of the function makes sense. We generally adopt the conventions that square roots of negative real numbers are not allowed, nor is division by zero.

- Example: Find the domains of the functions $g(x) = \sqrt{x+1}$ and $h(x) = \dfrac{1}{x^2 - 1}$.
  - For $g$, the values of $x$ in the domain are those which do not require taking the square root of a negative number. We require $x + 1 \geq 0$, which is the same as saying $\boxed{x \geq -1}$. We can also write the domain as the interval $\boxed{[-1, \infty)}$.
  - For $h$, the values of $x$ in the domain are those which do not require dividing by zero. We require $x^2 - 1 \neq 0$, which will be true whenever $x$ is not 1 or $-1$. Thus, the domain is $\boxed{\text{all real } x \neq \pm 1}$, which can be written as the union of intervals $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$.

- The notation $f(g(x))$ is used to symbolize the result of applying $f$ to the value $g(x)$: this is called function composition, and is well-defined provided that the range of $g$ is a subset of the domain of $f$. We use the notation $f \circ g$ to refer to the composite function itself, so that $(f \circ g)(x) = f(g(x))$.

- Example: Let $f(x) = x^2$ and $g(x) = 2x + 1$. Find $f \circ g$ and $g \circ f$.
  - We have $(f \circ g)(x) = f(g(x)) = f(2x + 1) = \boxed{(2x + 1)^2 = 4x^2 + 4x + 1}$.
  - Meanwhile, $(g \circ f)(x) = g(f(x)) = g(x^2) = \boxed{2x^2 + 1}$.
- In the example above, notice that the composition depends on the order. In general, it will be the case that $f \circ g$ and $g \circ f$ are *completely unrelated* functions!

- Definition: A function $f$ is one-to-one (or injective) if $f(a) = f(b)$ implies $a = b$, or equivalently, if $a \neq b$ then $f(a) \neq f(b)$. In other words, $f$ is one-to-one if unequal elements in the domain are sent to unequal elements in the range.
  - For functions whose domain and range are the real numbers, one-to-one functions satisfy the "horizontal line test": a horizontal line can intersect the graph of the function at most once.

- Definition: A one-to-one function $f(x)$ has an inverse function $f^{-1}(x)$ defined so that $f^{-1}(f(x)) = x$ for every $x$ in the domain of $f$, and $f(f^{-1}(y)) = y$ for every $y$ in the range of $f$.
  - To compute the inverse function of $f$, simply solve the equation $y = f(x)$ for $x$ in terms of $y$: this will give $x = f^{-1}(y)$.

- Example: Verify that the function $h(x) = 3x - 2$ is one-to-one and find its inverse function.
  - To show that $h$ is one-to-one, notice that $h(a) = h(b)$ is the same as $3a - 2 = 3b - 2$, and this can easily be rearranged to obtain $a = b$.
  - To find the inverse function, we want to solve $y = 3x - 2$ for $x$ in terms of $y$. We obtain $x = \dfrac{y+2}{3}$, so $\boxed{h^{-1}(y) = \dfrac{y+2}{3}}$.
  - Conceptually, $h$ multiplies its argument by 3 and then subtracts 2, so its inverse function necessarily reverses these operations, in the opposite order: namely, $h^{-1}$ first adds 2 and then divides its argument by 3.

- A function that is not one-to-one does not have a well-defined inverse function, because there will be an ambiguity somewhere: such a function must send two values in its domain to the same value in its range, but then this causes difficulties if we attempt to define an inverse function.
  - The way to get around this ambiguity problem is to narrow the domain of the function: if we restrict the domain so as to make the new restricted function one-to-one on the smaller domain, we can define an inverse function on that restricted domain.

- Example: Consider trying to define an inverse function for the function $f(x) = x^2$ defined for all real numbers $x$.
  - We have $f(2) = 2^2 = 4$, and so $f^{-1}$ should have $f^{-1}(4) = 2$. But it is also the case that $f(-2) = (-2)^2 = 4$, and so we should also have $f^{-1}(4) = -2$. This is a problem: $f^{-1}(4)$ cannot have two different values.
  - We can get around this problem by restricting the domain of $f$. Specifically, if we work with the function $g(x) = x^2$ defined only for $x \geq 0$, then $g$ does have an inverse, namely $g^{-1}(x) = \sqrt{x} = x^{1/2}$, the nonnegative square root of $x$.
  - By removing negative numbers from the domain of $f$, we have made a function $g$ that is one-to-one and that does possess an inverse function.

## 0.3 Algebra and Inequalities

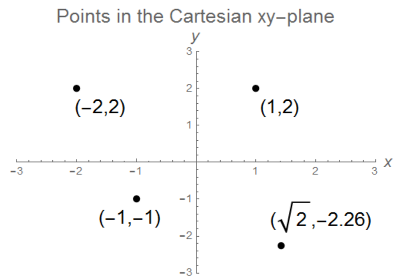
- One of the central properties of the real numbers is that they possess an ordering, and we can compare any two real numbers under this ordering: for any two real numbers $a$ and $b$, either $a < b$ ($a$ is less than $b$), $a = b$ ($a$ equals $b$), or $a > b$ ($a$ is greater than $b$).
  - Note that the statement $a > b$ ($a$ is greater than $b$) is the same as $b < a$ ($b$ is less than $a$).
  - A statement such as $3 < 7$ or $x + 1 > -3$ is called an <u>inequality</u>.
  - We also use the symbols $a \le b$ ($a$ is less than or equal to $b$) and $a \ge b$ ($a$ is greater than or equal to $b$) as shorthand to include the possibilities that $a = b$.
- Here are some fundamental properties of inequalities:
  - Adding and subtracting preserve inequalities: for any $a, b, c$, if $a < b$ then $a + c < b + c$ and $a - c < b - c$.
  - Multiplying by a positive constant preserves inequalities, and multiplying by a negative constant reverses them: for any $a, b, c$, if $a < b$ and $c > 0$ then $ac < bc$, and if $c < 0$ then $ac > bc$.
  - Similar results hold for the non-strict inequalities: if $a \le b$ then $a + c \le b + c$ and $a - c \le b - c$, if $a \le b$ and $c > 0$ then $ac \le bc$, if $a \le b$ and $c < 0$ then $ac \ge bc$.
  - In particular, $ab = 0$ precisely when at least one of $a$ and $b$ is zero, $ab > 0$ precisely when $a$ and $b$ have the same sign, and $ab < 0$ precisely when $a$ and $b$ have opposite signs.
  - Taking reciprocals reverses inequalities of numbers with the same sign: for any $a, b$ both positive or both negative, if $a < b$ then $\frac{1}{b} < \frac{1}{a}$.
- Using the basic properties, we can solve inequalities (i.e., characterize all the values satisfying an inequality) in the same way that we can solve equations.
- <u>Example</u>: Solve the inequality $-3x - 5 < 4$, and express the answer in interval notation.
  - First, we add 5 to both sides to obtain $-3x < 9$.
  - Now multiplying both sides by $-\frac{1}{3}$ yields $x > -3$ (note that the direction of the inequality reverses, because we multiplied by a negative number).
  - As an interval, we obtain the answer $\boxed{(-3, \infty)}$.
- <u>Example</u>: Solve the inequality $\frac{6}{3 - x} \ge -1$, and express the answer in interval notation.
  - First, if $3 - x$ is zero, then the quantity $\frac{6}{3 - x}$ is undefined, so $x \ne 3$.
  - Next, note that if $3 - x$ is positive (equivalently, if $x < 3$), then $\frac{6}{3 - x}$ is also positive, and so will automatically be greater than $-1$.
  - Finally, if $3 - x$ is negative, then multiplying both sides by $3 - x$ yields $6 \le (-1)(3 - x)$, where the inequality reversed direction since $3 - x$ is negative. Multiplying out yields $6 \le -3 + x$, which is the same as $x \ge 9$.
  - Therefore, the values satisfying the inequality are any $x$ with $x < 3$ or $x \ge 9$. In interval notation, we obtain $\boxed{(-\infty, 3) \cup [9, \infty)}$.
- We can also solve some basic inequalities involving quadratic functions.
  - Recall that the quadratic formula says the values of $x$ satisfying $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- <u>Example</u>: Find all values of $x$ satisfying the inequality $x^2 - 2x < 3$.

- Subtracting 3 from both sides yields $x^2 - 2x - 3 < 0$, and factoring the left-hand side yields $(x-3)(x+1) < 0$.
- The given quantity will be less than zero precisely when one term is positive and the other is negative.
- Since $x - 3 < x + 1$ we see that the inequality holds precisely when $\boxed{-1 < x < 3}$.

- An important function that often shows up in inequalities is the absolute value function:
- Definition: The absolute value of $x$, denoted $|x|$, is defined as $|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.
  - The absolute value function is an example of a piecewise-defined function, defined using different formulas on different parts of its domain.
  - Examples: $|4| = 4$, $|-5| = 5$, $|0| = 0$, $|\pi - 4| = 4 - \pi$.
  - Geometrically, the absolute value of $x$ represents the distance from $x$ to 0 along the real line, and $|x - y|$ represents the distance from $x$ to $y$.
- Here are some fundamental properties of absolute values:
  - The absolute value is multiplicative: for any $a$ and $b$, $|ab| = |a| \cdot |b|$. If $b \neq 0$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.
  - The triangle inequality: for any $a$ and $b$, $|a + b| \leq |a| + |b|$.
  - If $a > 0$, the statement $|x| < a$ is equivalent to $-a < x < a$, and $|x| > a$ is equivalent to $x < -a$ or $x > a$.
- Example: Solve the equation $\left|x^2 + 2x - 4\right| = 4$.
  - The given equation is equivalent to $x^2 + 2x - 4 = 4$ or $x^2 + 2x - 4 = -4$.
  - The first equation is the same as $x^2 + 2x - 8 = 0$, whose solutions by the quadratic formula are $x = \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm 6}{2} = -4, 2$.
  - The second equation is the same as $x^2 + 2x = 0$, which factors as $x(x + 2) = 0$, so that $x = -2, 0$.
  - Thus, we obtain $x = \boxed{-4, -2, 0, 2}$.
- Example: Find all values of $x$ satisfying the inequality $|2x - 5| \leq 3$.
  - Note that $|2x - 5| \leq 3$ is equivalent to $-3 \leq 2x - 5 \leq 3$.
  - Adding 5 everywhere yields $2 \leq 2x \leq 8$, and dividing by 2 yields $\boxed{1 \leq x \leq 4}$.
- Example: Solve the inequality $|2 - 3x| > 4$, and express the answer in interval notation.
  - Note that $|2 - 3x| = |3x - 2|$, so the inequality is the same as $|3x - 2| > 4$.
  - This inequality is in turn equivalent to $3x - 2 > 4$ or $3x - 2 < -4$.
  - We obtain $3x > 6$ or $3x < -2$, which are equivalent to $x > 2$ or $x < -\frac{2}{3}$. In interval notation, this is $\boxed{(-\infty, -\frac{2}{3}) \cup (2, \infty)}$.

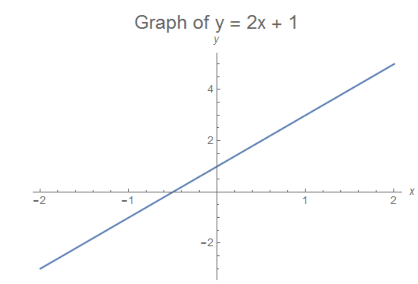
## 0.4 Coordinate Geometry and Graphs

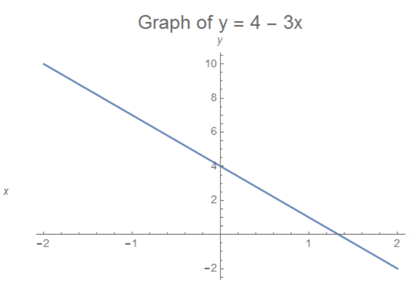
- We now turn our attention to coordinate geometry in the Cartesian $xy$-plane. We represent points in the plane using a pair of coordinates $(x, y)$, which denotes the point which is a horizontal distance $x$ and a vertical distance $y$ from the origin (0,0):

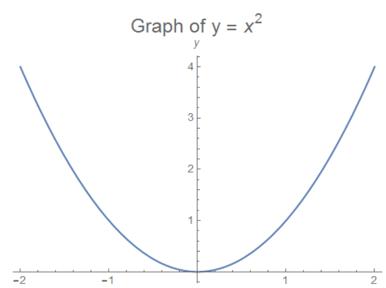


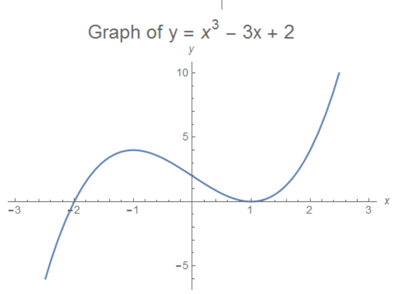

### 0.4.1 Graphs of Functions

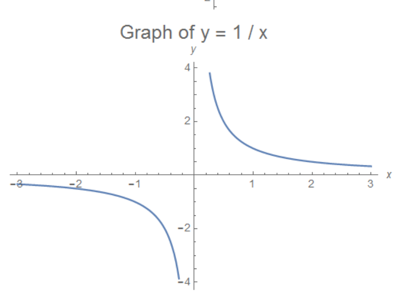
- We can graphically represent functions using coordinate geometry: if $f(x)$ is a function, we can "graph" the function by drawing all of the points $(x, f(x))$ in the plane.
  - We often describe a graph in the form $y = f(x)$, though we can also describe the points on a graph using more general relations, like $3x + 2y = 5$, or $x^2 + y^2 = 1$.
- Here are a few graphs of functions:

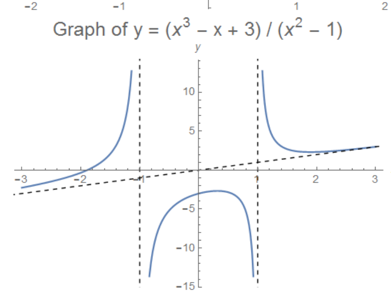




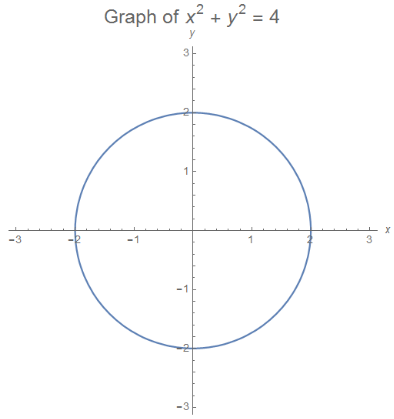


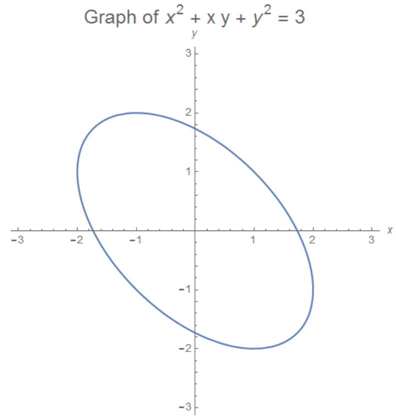


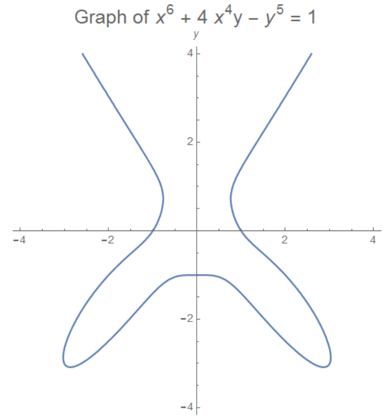





- Graphs do not need to be graphs of functions of the form $y = f(x)$: we can plot the set of points $(x, y)$ satisfying any relation, not just one of the form $y = f(x)$. Here are a few examples:

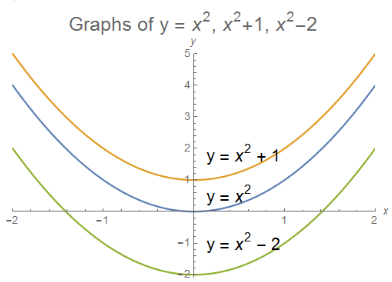


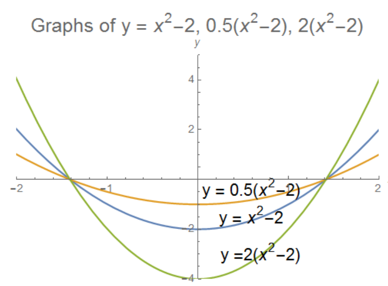




- There are a number of simple ways to transform the graph of a function using function composition. Here is a summary of such transformations:
  - If $k > 0$, the graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted up by $k$ units, and the graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted down by $k$ units.
  - If $k > 0$, the graph of $y = f(x + k)$ is the graph of $y = f(x)$ shifted left by $k$ units, and the graph of $y = f(x - k)$ is the graph of $y = f(x)$ shifted right by $k$ units.
  - If $k > 0$, the graph of $y = kf(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of $k$.
  - If $k > 0$, the graph of $y = f(x/k)$ is the graph of $y = f(x)$ stretched horizontally by a factor of $k$.
  - The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected vertically (through the $x$-axis), and the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected horizontally (through the $y$-axis).

- Example: Compare the graphs of $y = x^2$, $y = x^2 + 1$, $y = x^2 - 2$, $y = 2(x^2 - 2)$, and $y = \frac{1}{2}(x^2 - 2)$.
  - The graph of $y = x^2 + 1$ is the graph of $y = x^2$ translated up by 1 unit, and the graph of $y = x^2 - 2$ is the graph of $y = x^2$ translated down by 2 units:



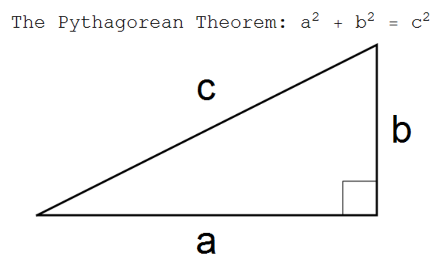




  - Likewise, the graph of $y = 2(x^2 - 2)$ is the graph of $y = x^2 - 2$ scaled vertically by a factor of 2, and the graph of $y = \frac{1}{2}(x^2 - 2)$ is the graph of $y = x^2 - 2$ scaled vertically by a factor of $\frac{1}{2}$ (i.e., compressed by a factor of 2).

- Example: In terms of $f$, find the function $h(x)$ whose graph $y = h(x)$ is obtained by first scaling the graph of $y = f(x)$ vertically by a factor of 2 and horizontally by a factor of 3, then translating left by 4 units and down by 1 unit.
  - The graph of $y = 2f(x/3)$ is the graph of $y = f(x)$ scaled vertically by a factor of 2 and horizontally by a factor of 3.
  - The graph of $y = g(x + 4) - 1$ is the graph of $y = g(x)$ translated left by 4 units and down by 1 unit.
  - If we then take $g(x) = 2f(x/3)$, this will compose the transformations in the proper order.
  - Thus, the desired function $h(x)$ is $h(x) = g(x + 4) - 1 = \boxed{2f\left(\frac{x+4}{3}\right) - 1}$.

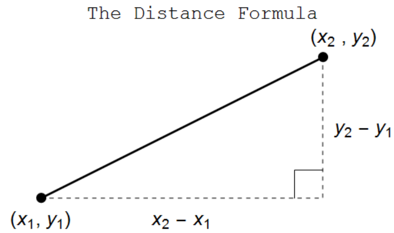
### 0.4.2 Lines and Distances

- The most basic graph is a line, whose most general equation has the form $ax + by = d$ for some constants $a, b, d$.
  - Examples of equations of lines: $x + y = 1$, $y = 3x + 2$, $x = \frac{1}{5}y - \pi$, $x = 4$.
  - A line passing through the two points $(x_1, y_1)$ and $(x_2, y_2)$ has slope $m = \frac{y_2 - y_1}{x_2 - x_1}$. The fundamental property of lines is that this ratio has the same value for any two points on the line.
  - The slope of a horizontal line is 0, and the slope of a vertical line is undefined (though it is often taken to be $\infty$).
  - Two lines are parallel if and only if their slopes are equal.
  - Two lines are perpendicular if and only if their slopes have product $-1$ (where we interpret $0 \cdot \infty = -1$ in this setting).
- There are two other common forms for the equation for a line:
  - Slope-intercept form: $y = mx + b$, where $m$ is the slope and $b$ is the $y$-intercept.
  - Point-slope form: $y - y_0 = m(x - x_0)$, where $m$ is the slope and $(x_0, y_0)$ is any point on the line.
  - Note that neither of these two forms can describe a vertical line, which has the special form $x = a$.
- Example: Find an equation for the line through $(1, 4)$ and $(3, 7)$.
  - To describe a line, we need a point on the line, and the slope.
  - The slope is $m = \frac{7-4}{3-1} = \frac{3}{2}$, and a point on the line is $(1, 4)$.
  - Hence, an equation in point-slope form is $\boxed{y - 4 = \frac{3}{2}(x - 1)}$.
  - We could also put this in slope-intercept form as $\boxed{y = \frac{3}{2}x + \frac{5}{2}}$.
- Example: Find an equation for the line through $(2, 5)$ perpendicular to the line $3x - 2y = 7$.
  - To describe a line, we need a point on the line, and the slope.
  - The line $3x - 2y = 7$ is the same as $y = \frac{3}{2}x - \frac{7}{2}$, which has slope $3/2$.
  - Hence the desired line has slope $\frac{-1}{3/2} = -\frac{2}{3}$.
  - Using point-slope, we obtain the equation $\boxed{y - 5 = -\frac{2}{3}(x - 2)}$.
- To discuss distances, we recall a central fact from geometry:
- Pythagorean Theorem: A right triangle with legs $a$ and $b$ and hypotenuse $c$ has $a^2 + b^2 = c^2$.




  - Three right triangles that show up frequently are the $1 - 1 - \sqrt{2}$, the $1 - \sqrt{3} - 2$, and the $3 - 4 - 5$.

- Distance Formula: The distance between $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
  - This formula is simply an application of the Pythagorean Theorem, where the leg $a$ is the distance $x_2 - x_1$ between the $x$-coordinates and the leg $b$ is the distance $y_2 - y_1$ between the $y$-coordinates:



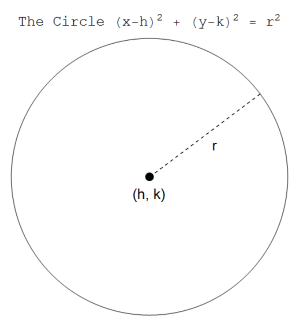

  - Example: The distance between $(1, 2)$ and $(4, 3)$ is $\sqrt{(4-1)^2 + (3-2)^2} = \boxed{\sqrt{10}}$.

- Definition: Given two points $(x_1, y_1)$ and $(x_2, y_2)$ in the plane, their midpoint is the point lying on the line segment between them that divides the segment into two pieces of equal length, and has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

- Example: Find an equation for the line passing through the points $(1, 1)$ and $(3, 5)$. Then verify that the midpoint lies on the line, and that it is equidistant from the two points.
  - Per the formula, the midpoint has coordinates $\left(\frac{1+3}{2}, \frac{1+5}{2}\right) = (2, 3)$.
  - The line itself has slope $\frac{5-1}{3-1} = 2$, so its equation is $y - 1 = 2(x - 1)$. We can see that the midpoint indeed lies on the line.
  - Finally, the distance from $(1, 1)$ to the midpoint is $\sqrt{1^2 + 2^2} = \sqrt{5}$, which is also the distance from $(3, 5)$ to the midpoint.

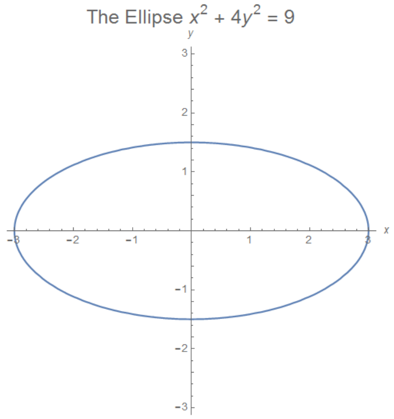
### 0.4.3 Circles and Conic Sections

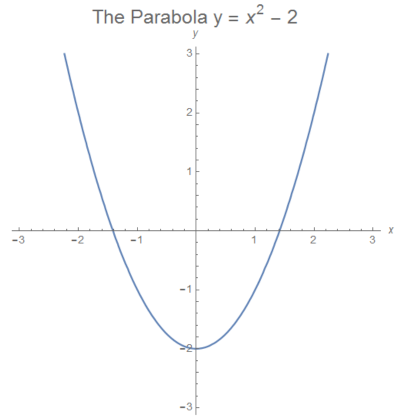
- Another important graph is the circle: geometrically, a circle is the set of points a fixed distance $r$ (called the radius) from a center point $(h, k)$. From the distance formula, we see that the equation of the circle of radius $r$ centered at $(h, k)$ is $(x - h)^2 + (y - k)^2 = r^2$:

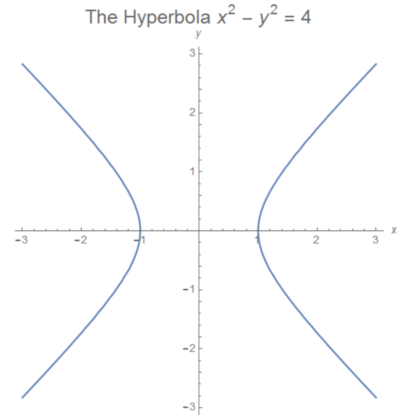



  - The circumference of a circle of radius $r$ is $2\pi r$ and the area is $\pi r^2$.

- A more general class of graphs is the conic sections, which have the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. They are the graphs of degree-2 equations in $x$ and $y$, in the same way that lines are the graphs of degree-1 equations in $x$ and $y$. Conic sections come in three kinds: ellipses (including circles), parabolas, and hyperbolas.



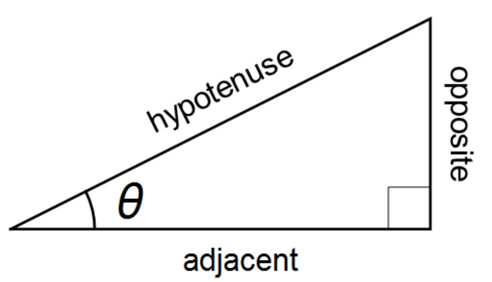

- An ellipse is the set of points whose sum of distances to two other points (the foci) is a fixed value. After an appropriate rotation and recentering of the coordinate axes, an ellipse can be put into the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- A parabola is the set of points whose distance from a fixed point (the focus) is equal to the distance from a fixed line (the directrix). After an appropriate rotation of the coordinate axes, a parabola can be put into the form $y = ax^2 + bx + c$.
- A hyperbola is the set of points whose difference of distances to two other fixed points (the foci) is a fixed value. After an appropriate rotation and recentering of the coordinate axes, a hyperbola can be put into the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- The conic sections are so named because they are the possible (nondegenerate) curves of intersection of a plane and a double-cone in 3-dimensional space. They have a number of useful applications and properties, which we will not detail here.

## 0.5 Trigonometry

- First, we remark that we always measure angles in radians: a right angle (90 degrees) is $\frac{\pi}{2}$ radians, and a full circle (360 degrees) is $2\pi$ radians.
  - We cannot really justify the purpose of using radians for angle measure at the moment: ultimately, the reason is that radians are the most natural measure for angles in calculus, and using radians will substantially simplify matters later on.

### 0.5.1 Trigonometric Functions

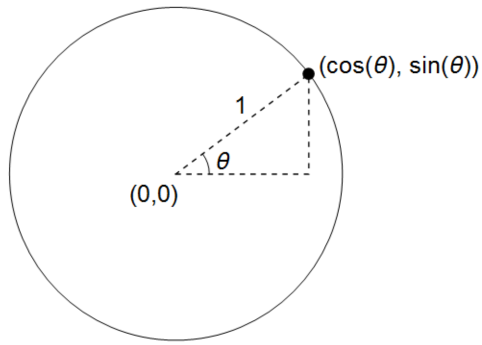
- The three basic trigonometric functions are sine, cosine, and tangent. They are defined as ratios between pairs of sides in a right triangle: explicitly, in a right triangle with an acute angle $\theta$, $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$.




  - The definitions using an acute triangle provide values for $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for any $0 \leq \theta \leq \pi/2$.

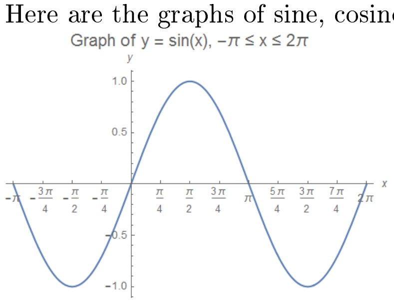
- To generalize the trigonometric functions to have a larger domain, we use the unit circle: per the definitions above, if we draw a ray from the origin making an angle $\theta$ with the positive $x$-axis, the ray will intersect the unit circle $x^2 + y^2 = 1$ at the point $(\cos(\theta), \sin(\theta))$.

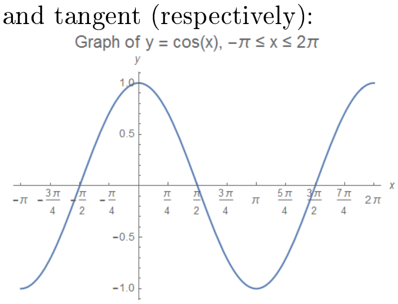
The Unit Circle $x^2 + y^2 = 1$

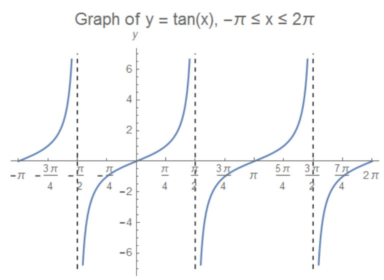



- We then define $\cos(\theta)$ to be the $x$-coordinate of the intersection point for an arbitrary angle $\theta$, and $\sin(\theta)$ to be the $y$-coordinate.

- Here are the graphs of sine, cosine, and tangent (respectively):




  - The ranges of sine and cosine are $[-1, 1]$ and the range of tangent is $(-\infty, \infty)$.
  - Sine and tangent are odd functions, meaning that $\sin(\theta) = -\sin(-\theta)$ and similarly for tangent. Cosine is an even function, meaning that $\cos(\theta) = \cos(-\theta)$.
  - Sine and cosine are periodic with period $2\pi$: thus, $\sin(\theta + 2\pi) = \sin(\theta)$ for any $\theta$, and similarly for cosine.
  - Tangent is periodic with period $\pi$: thus, $\tan(\theta + \pi) = \tan(\theta)$ for any $\theta$.
  - Note that $\tan(\pi/2)$ is undefined. We can see that as $\theta$ approaches $\pi/2$ from below, the value $\tan(\theta)$ goes to $+\infty$, and as $\theta$ approaches $\pi/2$ from above, the value $\tan(\theta)$ goes to $-\infty$.
  - The graph of $y = \tan(x)$ has a vertical asymptote at $x = \pi/2$. Since tangent is periodic, the same behavior occurs for $x = 3\pi/2$, $5\pi/2$, and so forth.

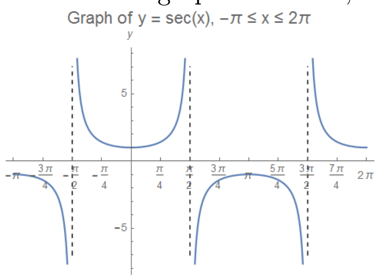
- There are a number of "special angles", where the values of the trigonometric functions are easy to calculate using geometry. Here is a table of commonly used values:

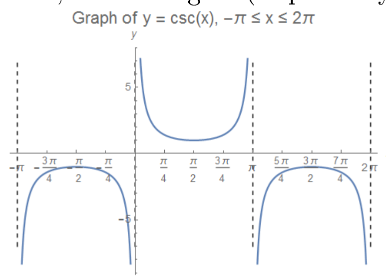
| $\theta$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $5\pi/6$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin(\theta)$ | 0 | $1/2$ | $\sqrt{2}/2$ | $\sqrt{3}/2$ | 1 | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | 0 |
| $\cos(\theta)$ | 1 | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | 0 | $-1/2$ | $-\sqrt{2}/2$ | $-\sqrt{3}/2$ | $-1$ |
| $\tan(\theta)$ | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | undef. | $-\sqrt{3}$ | $-1$ | $-1/\sqrt{3}$ | 0 |

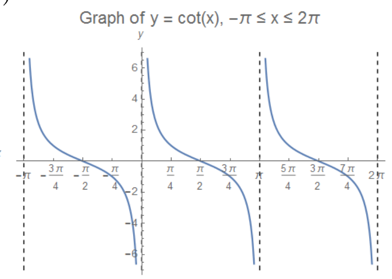
  - Using the table above and the periodicity and even/odd relations, one can find sines, cosines, and tangents of other angles.
  - For example, $\sin(11\pi/3) = -\sin(\pi/3) = \boxed{-\dfrac{\sqrt{3}}{2}}$, and $\cos(11\pi/6) = \cos(\pi/6) = \boxed{\dfrac{1}{2}}$.

- There are three other trigonometric functions that are used (though less frequently): secant, cosecant, cotangent.
  - They are defined as $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, and $\cot(\theta) = \dfrac{1}{\tan(\theta)}$.

- Here are the graphs of secant, cosecant, and cotangent (respectively):








- Cosecant and cotangent are odd functions, and secant is an even function.
- Secant and cosecant have period $2\pi$. Cotangent has period $\pi$.
- All of the functions have vertical asymptotes: secant has asymptotes in the same places as tangent, while cosecant and cotangent have asymptotes at integer multiples of $\pi$.

### 0.5.2 Trigonometric Identities

- There are many trigonometric identities. The most important identities are the following three, which hold for any angles $\theta$ and $\varphi$:
  - Addition formula for sine: $\sin(\varphi + \theta) = \sin(\varphi)\cos(\theta) + \cos(\varphi)\sin(\theta)$.
  - Addition formula for cosine: $\cos(\varphi + \theta) = \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta)$.
  - Pythagorean identity: $\sin^2(\theta) + \cos^2(\theta) = 1$.
  - We remark that it is standard notation to write $\sin^2(\theta)$ to mean $[\sin(\theta)]^2$, the square of $\sin(\theta)$, and more generally $\sin^k(\theta)$ denotes the $k$th power of $\sin(\theta)$ whenever $k$ is a positive integer.
- Using the results above, one can obtain a number of others, such as the following:
  - Addition formula for tangent: For any angles $\theta$ and $\varphi$, $\tan(\varphi + \theta) = \dfrac{\tan(\varphi) + \tan(\theta)}{1 - \tan(\varphi)\tan(\theta)}$.
  - Double-angle formulas: For any angle $\theta$,

$$\begin{aligned} \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \\ \tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}. \end{aligned}$$

  - Half-angle formulas: For any angle $0 \le \theta \le \pi$,

$$\begin{aligned} \sin(\theta/2) &= \sqrt{\frac{1 - \cos(\theta)}{2}} \\ \cos(\theta/2) &= \sqrt{\frac{1 + \cos(\theta)}{2}}. \end{aligned}$$

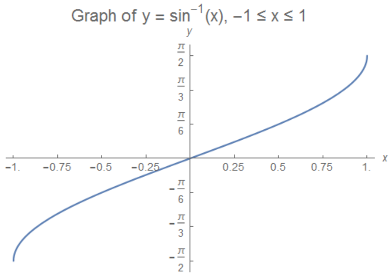
  - (Other) Pythagorean identities: For any angle $\theta$, $\tan^2(\theta) + 1 = \sec^2(\theta)$ and $1 + \cot^2(\theta) = \csc^2(\theta)$.
- The primary use of trigonometry in applications is to triangle measurement (indeed, the word "trigonometry" is Greek for "triangle measurement"), because the trigonometric functions relate angle measurements to side lengths in general triangles. Here are two fundamental results:
  - Law of sines: In triangle ABC, $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$, where $a$ = length of $BC$, $b$ = length of $AC$, $c$ = length of $AB$.
  - Law of cosines: In triangle ABC, $c^2 = a^2 + b^2 - 2ab\cos(C)$, where $a$ = length of $BC$, etc.

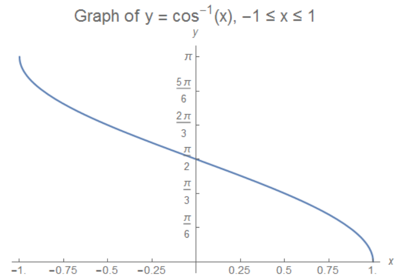
- Most problems in basic trigonometry can be solved by using one or more of the basic identities.
- <u>Example</u>: If $\tan(\theta) = \frac{2}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\sec(\theta)$, $\cos(\theta)$, $\sin(\theta)$, and $\sin(2\theta)$.
  - We first look for a relation between secant and tangent, which we can see is given by the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$.
  - Plugging in the given value yields $\sec^2(\theta) = 1 + \frac{4}{25} = \frac{29}{25}$, so $\sec(\theta) = \pm\frac{\sqrt{29}}{5}$. Since $\pi < \theta < \frac{3\pi}{2}$, and secant is negative on this interval, we conclude that $\sec(\theta) = \boxed{-\frac{\sqrt{29}}{5}}$.
  - Next, we have $\cos(\theta) = \frac{1}{\sec(\theta)} = \boxed{-\frac{5}{\sqrt{29}}}$.
  - To find sine, we use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to see that $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - \frac{25}{29} = \frac{4}{29}$.
  - Therefore, $\sin(\theta) = \pm\frac{2}{\sqrt{29}}$. Again, since $\pi < \theta < \frac{3\pi}{2}$, and sine is negative on this interval, we conclude that $\sin(\theta) = \boxed{-\frac{2}{\sqrt{29}}}$.
  - Finally, we can use the double-angle identity to write $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \boxed{\frac{20}{29}}$.
- <u>Example</u>: Find all angles $0 \le \theta \le 2\pi$ such that $\sin(\theta) + \cos(2\theta) = 1$.
  - We would like to try to write the given expression in terms of a single trigonometric function. We can do this using the double angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$.
  - Plugging in yields $\sin(\theta) + 1 - 2\sin^2(\theta) = 1$, which is the same as $\sin(\theta) - 2\sin^2(\theta) = 0$.
  - Factoring the left-hand side yields $\sin(\theta) \cdot [1 - 2\sin(\theta)] = 0$. The given expression will be zero precisely when $\sin(\theta) = 0$ or when $1 - 2\sin(\theta) = 0$, which is to say $\sin(\theta) = \frac{1}{2}$.
  - In the interval $[0, 2\pi]$, we know that $\sin(\theta) = 0$ when $\theta = 0, \pi, 2\pi$, and we know that $\sin(\theta) = \frac{1}{2}$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. So there are five solutions: $\theta = \boxed{0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi}$.

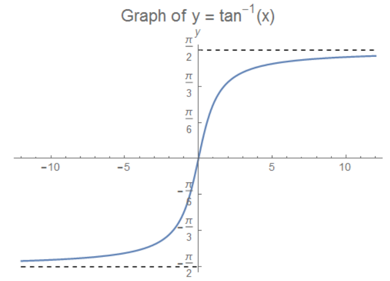
### 0.5.3 Inverse Trigonometric Functions

- None of the six standard trigonometric functions is one-to-one, so to define inverse functions, we restrict the domain of each function to an interval where it is one-to-one.
- Here are the standard definitions for arcsine (inverse sine), arccosine (inverse cosine), and arctangent (inverse tangent):
  - Arcsine: $\arcsin(x)$, also written $\sin^{-1}(x)$, is the inverse function of $\sin(x)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Its domain is $[-1, 1]$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
  - Arccosine: $\arccos(x)$, also written $\cos^{-1}(x)$, is the inverse function of $\cos(x)$ on the interval $[0, \pi]$. Its domain is $[-1, 1]$ and its range is $[0, \pi]$.
  - Arctangent: $\arctan(x)$, also written $\tan^{-1}(x)$, is the inverse function of $\tan(x)$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Its domain is $(-\infty, \infty)$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- Here are the graphs of arcsine, arccosine, and arctangent (respectively):



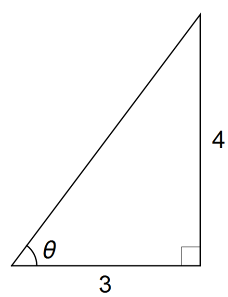




- Notational Warning: Despite the fact that $\sin^2(x)$ means the same thing as $[\sin(x)]^2$, the notation $\sin^{-1}(x)$ does NOT mean the same thing as $[\sin(x)]^{-1} = \csc(x)$. This kind of overload of notation, while confusing, is historical and used essentially everywhere. Some authors exclusively write $\arcsin(x)$ to denote the inverse sine function in order to avoid the possibility of confusion.

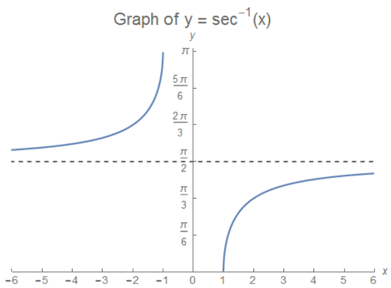
- We can compute some basic values using the definition: for example, $\sin^{-1}(1/2) = \boxed{\frac{\pi}{6}}$, from the list of special angles. We can find some others using geometry.

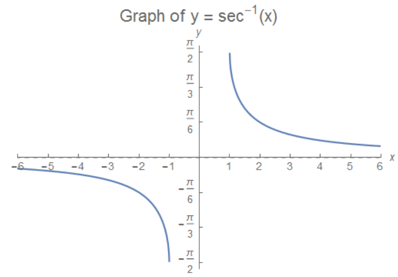
- Example: Find the exact value of $\cos(\tan^{-1}(4/3))$.
  - If $\theta = \tan^{-1}(4/3)$, then $\theta$ is the acute angle in a right triangle having opposite side of length 4 and adjacent side of length 3, as pictured below:

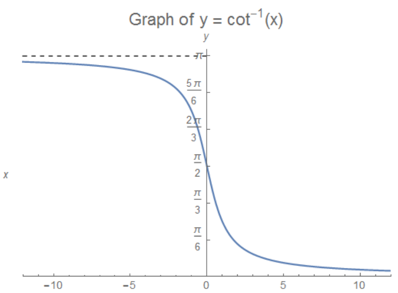



  - Then $\cos(\tan^{-1}(4/3)) = \cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, in this triangle.
  - The Pythagorean Theorem says that the hypotenuse has length $\sqrt{3^2+4^2} = 5$, so we see that $\cos(\theta) = \boxed{\frac{3}{5}}$.

- For the other three functions, there is some disagreement about the proper domain. We will not make much use of these, but here are the definitions (merely for the record):
  - Arcsecant: $\sec^{-1}(x)$ is the inverse function of $\sec(x)$ on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. Its domain is $(\infty, -1] \cup [1, \infty)$ and its range is $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.
  - Arccosecant: $\csc^{-1}(x)$ is the inverse function of $\csc(x)$ on $\left(-\frac{\pi}{2}, 0\right)$ and $\left(0, \frac{\pi}{2}\right)$. Its domain is $(\infty, -1] \cup [1, \infty)$ and its range is $\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$.
  - Arccotangent: $\cot^{-1}(x)$ is the inverse function of $\cot(x)$ on the interval $(0, \pi)$. Its domain is $(-\infty, \infty)$ and its range is $(0, \pi)$.
  - Here are graphs of the inverse secant, inverse cosecant, and inverse cotangent functions:



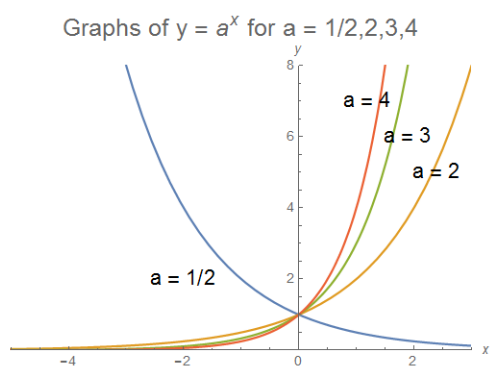





## 0.6 Exponentials and Logarithms

- Definition: The exponential function $a^x$ for $a > 0$ is motivated by the idea of repeated multiplication: if $n$ is a positive integer, $a^n$ is defined to be the result of multiplying $a$ by itself $n$ times: thus $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, and so forth.
  - For fractional powers, $\sqrt[q]{a} = a^{1/q}$ is defined to be the (nonnegative) real number whose $q$th power is $a$.
  - We can then define the exponential with any fractional exponent $a^{p/q}$ to be $(a^{1/q})^p$.
  - For negative powers, we also set $a^{-b} = \dfrac{1}{a^b}$.
  - Examples: We have $2^4 = \boxed{16}$, $\sqrt[3]{125} = \boxed{5}$, and $16^{-3/4} = \dfrac{1}{\left(16^{1/4}\right)^3} = \dfrac{1}{2^3} = \boxed{\dfrac{1}{8}}$.
  - For arbitrary real number exponents, we must resort to a limiting procedure. (We will omit the details.)
  - Integer powers of negative numbers (e.g., $(-2)^3 = -8$) are defined using repeated multiplication.
  - $0^b$ is zero for positive $b$ and is undefined for other $b$.
  - Non-integer powers of negative numbers (e.g., $(-2)^{-1/2}$) are not real numbers: the expressions can be given meaning using complex numbers, although there is some amount of ambiguity involved.

- Here are graphs of some exponential functions:




- In general, exponentials possess the following properties:
  - The domain of the function $f(x) = a^x$ for $a > 0$ is the real line $(-\infty, \infty)$, and the range for $a \neq 1$ is the interval $(0, \infty)$.
  - For any $x, a, b$, $x^{a+b} = x^a x^b$.
  - For any $x, y, a$, $(xy)^a = x^a y^a$.
  - For any $x, b, c$, $\left(x^b\right)^c = x^{bc}$.
  - For any $x, a$, $x^{-a} = \dfrac{1}{x^a}$.

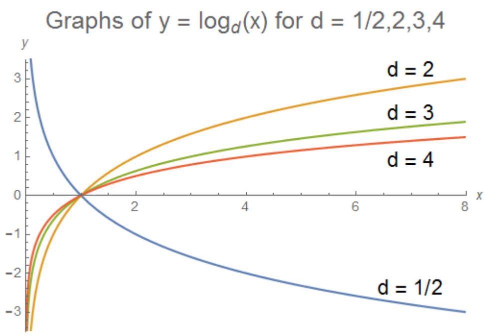
- Example: Express $(x^{2/3}y^{3/4})^2 \cdot (xy)^{-1}$ in the form $x^a y^b$.

- From the properties, $(x^{2/3}y^{3/4})^2 \cdot (xy)^{-1} = x^{4/3}y^{3/2} \cdot x^{-1}y^{-1} = \boxed{x^{1/3}y^{1/2}}$.

- For $b > 0$, it is easy to see that the exponential function $f(x) = b^x$ is one-to-one whenever $b \neq 1$, and therefore it has an inverse function.

- Definition: For $b > 0$ with $b \neq 1$, the general base-$b$ logarithm $\log_b x$ is defined to be the inverse function of the exponential function $f(x) = b^x$. Thus, the statement $y = \log_b x$ is equivalent to the statement that $x = b^y$.
  - The domain of $f(x) = \log_b x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.
  - Examples: $\log_5 125 = \boxed{3}$, since $125 = 5^3$, and $\log_3 \sqrt{27} = \boxed{\frac{3}{2}}$, since $\sqrt{27} = \sqrt{3^3} = 3^{3/2}$.

- Here are graphs of a few logarithm functions:




- For any positive $a, b, c, x, y$ with $b \neq 1$, the following properties hold:
  - The logarithm of 1 is always 0, in any base: $\log_b 1 = 0$.
  - Logarithms convert multiplication to addition: $\log_b [xy] = \log_b x + \log_b y$.
  - Exponents "drop down" in logarithms of powers: $\log_b [x^a] = a \log_b x$.
  - The "change of base" formula: $\dfrac{\log_a b}{\log_a c} = \log_c b$, also sometimes written as $\log_a b \cdot \log_b c = \log_a c$.

- Example: Write $\log_{10}(x^4y^3)$ in the form $a \log_{10}(x) + b \log_{10}(y)$.
  - From the properties, $\log_{10}(x^4y^3) = \log_{10}(x^4) + \log_{10}(y^3) = \boxed{4\log_{10}(x) + 3\log_{10}(y)}$.

- One of the chief uses of logarithms is to solve equations involving exponentials.

- Example: Solve the equation $7^{3x+4} = 3$ for $x$.
  - We take the logarithm to the base 7 of both sides: $\log_7(7^{3x+4}) = \log_7(3)$.
  - Since the logarithm and exponential are inverses, $\log_7(7^{3x+4}) = 3x + 4$.
  - Thus, $3x + 4 = \log_7(3)$, so $x = \boxed{\dfrac{\log_7(3) - 4}{3}} \approx -1.145$.

- Example: Find all real numbers $x$ for which $4^x - 2^{x+3} + 12 = 0$.
  - If we write $y = 2^x$, then $4^x = y^2$ while $2^{x+3} = 8y$, so the given equation is equivalent to $y^2 - 8y + 12 = 0$.
  - Factoring yields $(y - 2)(y - 6) = 0$, so $y = 2$ or $y = 6$.
  - Thus, $2^x = 2$ or $2^x = 6$, so the solutions are $x = \boxed{1, \log_2 6}$.

- Example: If $f(x) = 3^{4-x} + 2$, find the inverse function $f^{-1}(x)$.
  - We wish to solve $y = 3^{4-x} + 2$ for $x$ in terms of $y$.
  - First, we rewrite $y - 2 = 3^{4-x}$.

- Now taking the logarithm to the base 3 yields $\log_3(y-2) = \log_3(3^{4-x}) = 4-x$.
- Then $x = 4 - \log_3(y-2)$, meaning that $f^{-1}(y) = 4 - \log_3(y-2)$.
- Equivalently, this says $f^{-1}(x) = \boxed{4 - \log_3(y-2)}$.

- There is a particular logarithm base that is natural from the standpoint of calculus.
- Definition: The natural logarithm $\ln(x)$ is equal to $\log_e(x)$, whose base is the number $e \approx 2.718$.
  - The choice of this somewhat-strange number $e$ as our logarithm base seems arbitrary. In fact, this logarithm base, as we will see later, is by far the most natural choice in the context of calculus, in much the same way that radians are the most natural choice for measuring angles.
  - There are many different formulas for the number $e$. One of the simplest is $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$, where $n!$ is the factorial of $n$, defined as $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$.
  - Another formula for $e$ (which arises from its connection to continuously compounded interest) is as the value approached by the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ as $x$ grows arbitrarily large.
  - In general, all logarithms can be expressed in terms of the natural logarithm via the change-of-base formula: $\log_b a = \frac{\ln(a)}{\ln(b)}$.
- Example: Find the numerical value of $e^{4\ln 2 + \ln 3}$.
  - Observe that $4\ln(2) + \ln(3) = \ln(2^4) + \ln(3) = \ln(2^4 \cdot 3) = \ln(48)$.
  - Then $e^{4\ln 2 + \ln 3} = e^{\ln 48} = \boxed{48}$.
- Example: Find all real numbers $x$ for which $\ln(x) + \ln(3x+2) = 2\ln(x+2)$.
  - Notice that $\ln(x) + \ln(3x+2) = \ln[x(3x+2)] = \ln(3x^2 + 2x)$.
  - Similarly, $2\ln(x+2) = \ln[(x+2)^2] = \ln(x^2 + 4x + 4)$.
  - Thus, $\ln(3x^2 + 2x) = \ln(x^2 + 4x + 4)$.
  - Exponentiating both sides yields $3x^2 + 2x = x^2 + 4x + 4$, or equivalently $2x^2 - 2x - 4 = 0$.
  - Factoring yields $2(x+1)(x-2) = 0$, so the possible solutions are $x = -1$ and $x = 2$.
  - However, since $\ln(-1)$ is not a real number, $x = -1$ is not actually a solution to the equation.
  - We can see that $x = 2$ does work, however, so the only solution is $\boxed{x = 2}$.
- Example: Find the inverse of the function $f(x) = 4\ln(3x-2) + 1$.
  - We wish to solve $y = 4\ln(3x-2) + 1$ for $x$ in terms of $y$.
  - First, we isolate the logarithm term: we have $\frac{y-1}{4} = \ln(3x-2)$.
  - Now exponentiating both sides yields $e^{(y-1)/4} = e^{\ln(3x-2)} = 3x - 2$.
  - Then solving for $x$ yields $x = \frac{1}{3}\left[e^{(y-1)/4} + 2\right]$, meaning that $f^{-1}(y) = \frac{1}{3}\left[e^{(y-1)/4} + 2\right]$.
  - Equivalently, this says $f^{-1}(x) = \boxed{\frac{1}{3}\left[e^{(x-1)/4} + 2\right]}$.
- As a final remark, we note that logarithms of negative numbers and logarithms with a negative base, such as $\ln(-1)$ and $\log_{-2} 5$, are not real numbers.
  - These expressions can be given meaning using complex numbers, although there is some amount of ambiguity involved. We will not discuss the matter further at present.

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Well, you're at the end of my handout. Hope it was helpful.