Math 1341 - Final Exam Review #2

December 4, 2019

Final Exam Topics:

- Average rate of change, limit definition of derivative
- Computing derivatives (product/quotient/chain rules)
- Logarithmic, inverse, implicit differentiation
- Parametric curves and derivatives, velocity/speed/acceleration
- Related rates
- Minimum and maximum values, crit points + classification, increasing and decreasing behavior, concavity, inflection points
- L'Hôpital's rule
- Applied optimization
- Antiderivatives
- Riemann sums + definite integrals, Fund Thm of Calculus
- Evaluating definite and indefinite integrals, substitution
- Areas under and between curves

Problem 1

(Fa14, #10) A box with an open top is to be constructed with $600\,\mathrm{in}^2$ of material. The length of the base is to be twice its width. Find the dimensions that maximize the volume of the box.

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Answer: Width w, height h, length $l = 2w$, vol $V = lwh = 2w^2h$. Base has area $2w^2$, two sides have area wh, and the other two sides have area $2wh$. Hence total area is $2w^2 + 6wh$, so $2w^2 + 6wh = 600$, thus $h = \frac{300-w^2}{3w}$.

Then $V = 2w^2 \cdot \frac{300-w^2}{3w} = 200w - \frac{2}{3}w^3$ so $V' = 200 - 2w^2$ which is zero when $w = 10\,\mathrm{in}$. Sign diagram for V' shows $w = 10\,\mathrm{in}$ is a global max. So dimensions are $\boxed{w = 10\,\mathrm{in},\ l = 20\,\mathrm{in},\ h = \frac{20}{3}\,\mathrm{in}}$.

Problem 2

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Answer: The curve intersects the x-axis when $2x - x^2 = 0$ so that $x = 0, 2$. Then the desired area is

$$\int_0^2 (2x - x^2)\,dx = x^2 - \frac{1}{3}x^3\Big|_{x=0}^{2} = \boxed{\frac{4}{3}}.$$

Problem 3

(Fa14, #11a) Compute $\int (7+8x)^{49}\,dx$.

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Answer:

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Answer: Substituting $u = 7+8x$ with $du = 8\,dx$ yields

$$I = \int u^{49} \cdot \frac{1}{8}\,du = \frac{1}{400}u^{50} + C = \boxed{\frac{1}{400}(7+8x)^{50} + C}.$$

Interlude!

Problem 4

(Fa14, #13) Find the area bounded by $y = x^2 - 5x + 3$ and $y = -x^2 + x - 1$.

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(Fa14, #13) Find the area bounded by $y = x^2 - 5x + 3$ and $y = -x^2 + x - 1$.

Answer: The curves intersect when $x^2 - 5x + 3 = -x^2 + x - 1$ so that $2x^2 - 6x + 4 = 0$. Factoring gives $2(x-1)(x-2) = 0$ so intersection points are at $x = 1, 2$.

Testing at $x = 3/2$, or comparing the graphs, shows that $y = -x^2 + x - 1$ is the top curve and $y = x^2 - 5x + 3$ is the bottom curve.

Hence area is $\int_1^2 [\text{top} - \text{bottom}]\, dx = \int_1^2 (-2x^2 + 6x - 4)\, dx =$
$\left(-\frac{2}{3}x^3 + 3x^2 - 4x\right)\Big|_{x=1}^{2} = \boxed{\frac{1}{3}}$.

Problem 5

(Fa14, #11b) Compute $\displaystyle\int_{\pi}^{2\pi} \frac{3\sin x}{2+\cos x}\,dx$.

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Answer: Substitute $u = 2 + \cos x$ with $du = -\sin x\,dx$. Then $x = \pi$ corresponds to $u = 1$ and $x = 2\pi$ corresponds to $u = 3$, so then we obtain $\displaystyle I = \int_1^3 \frac{-3}{u}\,du = -\ln(u)\Big|_{x=1}^{3} = \boxed{-3\ln 3}$.

Interlude!

Problem 6

(Fa15, #10) Suppose we need to construct a coffee cup in the shape of a circular cylinder that holds 128π cubic centimeters. The cup has no top! The cost per square centimeter of material for the sides of the cup is 1 cent, and for the bottom of the cup the cost per square centimeter is 2 cents. Find the radius r and height h of the cup that minimizes the cost.

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Answer: Volume is $V = \pi r^2 h \text{ cm}^3$, so $r^2 h = 128$ hence $h = \frac{128}{r^2}$.
Area of sides is $2\pi r h \text{ cm}^2$, area of base is $\pi r^2 \text{ cm}^2$, so total cost is $C = 2\pi r h + 2\pi r^2 = 2\pi(128/r + r^2)$ cents.

So $C'(r) = 2\pi(-128/r^2 + 2r)$ which is zero for $r = 4 \text{ cm}$, and is the global min by C' sign diagram. So $\boxed{r = 4\,\text{cm} \text{ and } h = 8\,\text{cm}}$.

Problem 7

(Fa14, #7) Consider the function $f(x) = x^3 + \frac{1}{2}x^2$. Find $f(2)$ and $(f^{-1})'(10)$.

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(Fa14, #7) Consider the function $f(x) = x^3 + \frac{1}{2}x^2$. Find $f(2)$ and $(f^{-1})'(10)$.

Answer: Note that $f'(x) = 3x^2 + x$.

Clearly $f(2) = \boxed{10}$, meaning that $f^{-1}(10) = 2$.

Then by the inverse function differentiation formula,

$$(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(2)} = \boxed{\frac{1}{14}}.$$

Interlude!

Problem 8

(Fa14, #10) Find the area of the region bounded by the graphs of $f(x) = x^2 - x - 1$ and $g(x) = x + 2$.

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(Fa14, #10) Find the area of the region bounded by the graphs of $f(x) = x^2 - x - 1$ and $g(x) = x + 2$.

Answer: The curves intersect when $x^2 - x - 1 = x + 2$ so that $x^2 - 2x - 3 = 0$. Factoring gives $(x-3)(x+1) = 0$ so intersection points are at $x = -1, 3$.

Testing at $x = 0$, or comparing the graphs, shows that $y = x + 2$ is the top curve and $y = x^2 - x - 1$ is the bottom curve.

Hence area is $\int_{-1}^{3} [\text{top} - \text{bottom}]\, dx = \int_{-1}^{3} (-x^2 + 2x + 3)\, dx =$ $\left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{x=-1}^{3} = \boxed{\dfrac{32}{3}}$.

Problem 9

(Fa14, #12) Compute the midpoint Riemann sum for $f(x) = x^2$ for the partition of the interval $[-\frac{1}{2}, 1]$ into 3 subintervals of equal length.

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(Fa14, #12) Compute the midpoint Riemann sum for $f(x) = x^2$ for the partition of the interval $[-\frac{1}{2}, 1]$ into 3 subintervals of equal length.

Answer: The width of the subintervals is $\dfrac{1-(-1/2)}{3} = \dfrac{1}{2}$, and the subintervals are $[-\frac{1}{2}, 0]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$.

Then the Riemann sum is

$$RS_{\text{mid}} = f(-\frac{1}{4}) \cdot \frac{1}{2} + f(\frac{1}{4}) \cdot \frac{1}{2} + f(\frac{3}{4}) \cdot \frac{1}{2} = \boxed{\frac{11}{32}}.$$

Interlude!

Problem 10

(Sp17, #10c) Evaluate $\int_{-\pi}^{\pi} \sin(x)\cos^2(x)\,dx$.

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Answer: Substitute $u = \cos(x)$ so that $du = -\sin(x)\,dx$. Then $x = -\pi$ corresponds to $u = -1$ and $x = \pi$ also corresponds to $u = -1$, so the integral is $I = \int_{-1}^{-1} -u^2\,du = -\frac{1}{3}u^3\Big|_{u=-1}^{-1} = \boxed{0}$.

Problem 11

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Answer: The curves intersect when $5x = x^2 + 4$ so $x = 1, 4$.

Using a test point or comparing graphs shows that $y = 5x$ is the top curve and $y = x^2 + 4$ is the bottom curve for $1 \leq x \leq 4$.

Then the desired area is

$$\int_1^4 (5x - x^2 - 4)\, dx = \left(\frac{5}{2}x^2 - \frac{1}{3}x^3 - 4x\right)\Big|_{x=1}^{4} = \boxed{\frac{9}{2}}.$$

Interlude!

Problem 12

Evaluate $\int \tan^3 x \sec^2 x \, dx$.

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Answer:

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Answer: Substitute $u = \tan(x)$ so that $du = \sec^2(x)dx$. Then

$$I = \int u^2 \, du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\tan^3(x) + C}.$$

Problem 13

(Fa14, #11d) Compute $\int \frac{4t^4 - 3t + \sqrt[3]{t}}{t^2}\,dt$.

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Answer:

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(Fa14, #11d) Compute $\displaystyle\int \frac{4t^4 - 3t + \sqrt[3]{t}}{t^2}\,dt$.

Answer: Distribute the integrand to obtain

$$I = \int \left(4t^2 - \frac{3}{t} + t^{-5/3}\right) dt = \boxed{\frac{4}{3}t^3 - 3\ln t - \frac{3}{2}t^{-2/3}}.$$

End

Enjoy WeBWorK #12, and happy last day of fall classes!