Math 1341 - Final Exam Review #1

December 2, 2019

Final Exam Topics:

- Average rate of change, limit definition of derivative
- Computing derivatives (product/quotient/chain rules)
- Logarithmic, inverse, implicit differentiation
- Parametric curves and derivatives, velocity/speed/acceleration
- Related rates
- Minimum and maximum values, crit points + classification, increasing and decreasing behavior, concavity, inflection points
- L'Hôpital's rule
- Applied optimization
- Antiderivatives
- Riemann sums + definite integrals, Fund Thm of Calculus
- Evaluating definite and indefinite integrals, substitution
- Areas under and between curves

Problem 1

(Sp17, #1b) Find the derivative of $f(x) = \sqrt{3x - 2}$ at $x = 2$ using the definition of the derivative as a limit.

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Answer:

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Answer: By definition and the "multiply by the conjugate" trick,

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{4+3h}-2}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+3h}-2}{h} \cdot \frac{\sqrt{4+3h}+2}{\sqrt{4+3h}+2} = \lim_{h\to 0} \frac{4+3h-4}{h(\sqrt{4+3h}+2)}$$

$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{4+3h}+2)} = \lim_{h\to 0} \frac{3}{\sqrt{4+3h}+2} = \boxed{\frac{3}{4}}.$$

Problem 2

(Sp14, #1c) Find the equation of the tangent line to the graph of $f(x) = \dfrac{1}{3x+1}$ at $x = 0$.

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Answer: Since $f(0) = 1$, the line passes through $(0, 1)$.
Also, $f'(x) = \dfrac{-3}{(3x+1)^2}$ by the chain rule, so $f'(0) = -3$.
So the line passes through $(0, 1)$ and has slope -3, so its equation is $\boxed{y - 1 = -3(x - 0)}$, or equivalently, $\boxed{y = -3x + 1}$.

Interlude!

Problem 3

(Sp14, #2) Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$. Find the critical points of f and all local minima and local maxima (including the values of f), and also identify all intervals where f is increasing and where f is decreasing.

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Answer: We have $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2)$, so the critical numbers are $x = -1, 0, 2$.

Using test points gives a sign diagram $f' : \ominus \underset{-1}{|} \oplus \underset{0}{|} \ominus \underset{2}{|} \oplus$, so f has a $\boxed{\text{local max at } (0,3)}$ and $\boxed{\text{local mins at } (-1,-2),\ (2,-29)}$, and is $\boxed{\text{increasing on } (-1,0),(2,\infty)}$, $\boxed{\text{decreasing on } (-\infty,-1),(0,2)}$.

Problem 4

(Sp14, #3) Find all intervals where $f(x) = \frac{1}{4}x^4 - 6x^2 + 6$ is concave up and where it is concave down, and find all inflection points (include both x and y coordinates).

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Answer: We have $f''(x) = 3x^2 - 12 = 3(x-2)(x+2)$, so the potential inflection numbers are $x = -2, 2$.

Using test points gives a sign diagram $f'' : \oplus \underset{-2}{|} \ominus \underset{2}{|} \oplus$, so f is

$\boxed{\text{concave up on } ((-\infty, -2), (2, \infty)}$, $\boxed{\text{concave down on } (-2, 2)}$.

Since the concavity changes at both points, there are

$\boxed{\text{inflection points at } (-2, -14) \text{ and } (2, -14)}$.

Interlude!

Problem 5

(Sp14, #4ab) Find the derivatives of (a) $f(x) = x\sin(\sqrt{x})$ and (b) $g(x) = \log_4(2x+1) - 4^{2x+1} + \frac{1}{2}e^{2x}$.

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Answer: We have $f'(x) = \boxed{\sin(\sqrt{x}) + x\cos(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}}$ by the product and chain rules. Also,

$$g'(x) = \boxed{\frac{1}{\ln(4)} \cdot \frac{2}{2x+1} - 4^{2x+1}\ln(4) \cdot 2 + e^{2x}}$$ using the basic derivatives and the chain rule.

Problem 6

(Sp14, #4d) Let $f(x) = 2x^2 - 3x + 1$ where $x \geq 0$. Find the values of $f(2)$ and $(f^{-1})'(3)$.

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Answer: Note that $f(2) = \boxed{3}$ by plugging in.

The inverse function deriv. formula says $(f^{-1})'(3) = \dfrac{1}{f'\left[f^{-1}(3)\right]}$.

From the fact that $f(2) = 3$ we know $f^{-1}(3) = 2$, and so we get $(f^{-1})'(3) = \dfrac{1}{f'(2)} = \boxed{\dfrac{1}{5}}$ since $f'(x) = 4x - 3$.

Interlude!

Problem 7

(Sp14, #5abc) A particle's position in the plane is given by $p(t) = (3\cos(t), 3\sin(t))$. Find its velocity, speed, and acceleration at time t.

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Answer: The velocity is $p'(t) = \boxed{(-3\sin(t), 3\cos(t))}$.

The speed is $||p'(t)|| = \boxed{\sqrt{9\sin^2 t + 9\cos^2 t} = 3}$.

The acceleration is $a(t) = p''(t) = \boxed{(-3\cos(t), -3\sin(t))}$.

Problem 8

(Sp14, #6b) Find the limit $\lim_{x \to 2+} \dfrac{\ln(x-1)}{\sqrt{x-2}}$.

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Answer: As $x \to 2+$ the numerator and denominator go to 0 so this is a $\frac{0}{0}$ limit. By L'Hopital's rule,

$$L = \lim_{x \to 2+} \frac{1/(x-1)}{\frac{1}{2}(x-2)^{-1/2}} = \lim_{x \to 2+} \frac{2(x-2)^{1/2}}{x-1} = \boxed{0}.$$

Interlude!

Problem 9

(Sp14, #7) Find the point (x, y) on the graph of $y = \sqrt{x}$ nearest to the point $(3, 0)$. [Hint: Minimize the square of the distance.]

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Answer: To be on the graph, the point must be of the form $(x, \sqrt{x})$. The distance between this point and $(3, 0)$ is $\sqrt{(x-3)^2 + (\sqrt{x}-0)^2}$. It is much easier to minimize the square of the distance, which is
$f(x) = (x-3)^2 + (\sqrt{x}-0)^2 = x^2 - 6x + 9 + x = x^2 - 5x + 9.$

Then $f'(x) = 2x - 5$ which is zero when $x = 5/2$. Using a sign diagram we can see that this is the global minimum value for f, so the minimum distance occurs at the point $\boxed{(\frac{5}{2}, \sqrt{\frac{5}{2}})}$.

Problem 10

(Sp14, #8b) Compute $\displaystyle\int_0^{\pi/4} \cos(2t)\,dt$.

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Answer: Make the substitution $u = 2t$ with $du = 2dt$. Then $t = 0$ corresponds to $u = 0$ and $t = \pi/4$ corresponds to $u = \pi/2$, so the integral is $I = \int_0^{\pi/2} \cos(u) \cdot \frac{1}{2}\,du = \frac{1}{2}\sin(u)\Big|_{u=0}^{u=\pi/2} = \boxed{\frac{1}{2}}$.

Interlude!

Problem 11

(Sp14, #8c) Compute $\displaystyle\int_{e^2}^{e^3} \frac{\ln x}{x}\, dx$.

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Answer:

Problem 11

(Sp14, #8c) Compute $\displaystyle\int_{e^2}^{e^3} \frac{\ln x}{x}\,dx$.

Answer: Make the substitution $u = \ln x$ with $du = \dfrac{1}{x}\,dx$.

Then $x = e^2$ corresponds to $u = 2$ and $x = e^3$ corresponds to $u = 3$, so the integral is $I = \displaystyle\int_2^3 u\,du = \frac{1}{2}u^2\Big|_{u=2}^{3} = \boxed{\frac{5}{2}}$.

Problem 12

(Sp14, #8d) Compute $\int \frac{3w^3 - 2w^2 + 5}{w^3}\, dw$.

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Answer:

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(Sp14, #8d) Compute $\int \frac{3w^3 - 2w^2 + 5}{w^3}\, dw$.

Answer: Distribute the fraction to obtain

$$I = \int \left[3 - 2w^{-1} + 5w^{-3}\right]\, dw = \boxed{3w - 2\ln(w) - \frac{5}{2}w^{-2} + C}.$$

Problem 13

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Answer: The width of each subinterval is $\dfrac{6-(-4)}{5} = 2$ so the subintervals are $[-4,-2]$, $[-2,0]$, $[0,2]$, $[2,4]$, $[4,6]$. Then the midpoint Riemann sum is

$$RS_{\text{mid}} = f(-3)\cdot 2 + f(-1)\cdot 2 + f(1)\cdot 2 + f(3)\cdot 2 + f(5)\cdot 2 = \boxed{120}.$$

End

Enjoy WeBWorK #10, and I will see you on Monday for more review!