Math 1341 - Midterm 2 Review #2

November 18, 2019

Midterm 2 Topics:

- Related rates
- Minimum and maximum values, critical points + classification
- Increasing and decreasing behavior, concavity
- Rolle's theorem + mean value theorem
- L'Hôpital's rule
- Antiderivatives
- Riemann sums + properties of definite integrals
- Fundamental theorem of calculus
- Evaluating definite and indefinite integrals

(Differentiating integrals and substitution are not on the midterm, though they are fair game for the final.)

Problem 1

Find $\displaystyle\int \left(\frac{1}{\sqrt{1 - x^2}} + \frac{1}{1 + x^2} \right) dx$.

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Answer:

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Find $\displaystyle\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{1+x^2} \right) dx$.

Answer: By our basic integrals this is $\boxed{\sin^{-1}(x) + \tan^{-1}(x) + C}$.

Problem 2

Find the left-endpoint, midpoint, and right-endpoint Riemann sums for $f(x) = \sqrt{x}$ on $[0, 4]$ with 2 equal subintervals.

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Find the left-endpoint, midpoint, and right-endpoint Riemann sums for $f(x) = \sqrt{x}$ on $[0, 4]$ with 2 equal subintervals.

Answer: With 2 equal subintervals the intervals are $[0, 2]$ and $[2, 4]$. So we see

$$RS_{\text{left}} = f(0) \cdot 2 + f(2) \cdot 2 = \boxed{2\sqrt{2}} \approx 2.828,$$

$$RS_{\text{mid}} = f(1) \cdot 2 + f(3) \cdot 2 = \boxed{2 + 2\sqrt{3}} \approx 5.464,$$

$$RS_{\text{right}} = f(2) \cdot 2 + f(4) \cdot 2 = \boxed{4 + 2\sqrt{2}} \approx 6.828.$$

Problem 3

Evaluate $\lim_{x\to\infty} (x^2 + 3x)^{4/\ln(x)}$.

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Answer:

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Evaluate $\lim_{x\to\infty} (x^2 + 3x)^{4/\ln(x)}$.

Answer: This is an ∞^0 limit. Take the natural log to see $\ln L =$ $\lim_{x\to\infty} \ln\left[(x^2+3x)^{4/\ln(x)}\right] = \lim_{x\to\infty} \frac{4\ln(x^2+3x)}{\ln(x)}$.

Now apply L'Hôpital's rule to obtain $\ln L =$ $\lim_{x\to\infty} \frac{4(2x+3)/(x^2+3x)}{1/x} = \lim_{x\to\infty} \frac{4x(2x+3)}{x^2+3x} = \lim_{x\to\infty} \frac{8x^2+12x}{x^2+3x}$.

By another two applications of L'Hôpital (or just by comparing leading terms) we see that this limit is 8. So $\ln L = 8$ and $L = \boxed{e^8}$.

Interlude!

Problem 4

If $\int_1^3 f(x)\,dx = 4$ and $\int_3^4 f(x)\,dx = 5$, find $\int_1^4 [2f(x) + x]\,dx$.

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Answer:

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Answer: Using integration properties we see that
$\int_1^4 [2f(x) + x]\,dx = \int_1^4 2f(x)\,dx + \int_1^4 x\,dx$.

Then $\int_1^4 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx = 9$ and
$\int_1^4 x\,dx = \frac{1}{2}x^2\big|_{x=1}^4 = \frac{15}{2}$.

So the integral is $\int_1^4 [2f(x) + x]\,dx = 2 \cdot 9 + \frac{15}{2} = \boxed{\frac{51}{2}}$.

Problem 5

Allan has 12 feet of string. He uses some to form a square and the rest to form a right triangle with sides in the ratio 3:4:5. Find the maximum and minimum possible total areas of Allan's two shapes. [Hint: Take the triangle sides to be $3s$, $4s$, $5s$.]

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Allan has 12 feet of string. He uses some to form a square and the rest to form a right triangle with sides in the ratio 3:4:5. Find the maximum and minimum possible total areas of Allan's two shapes. [Hint: Take the triangle sides to be $3s$, $4s$, $5s$.]

Answer: If the triangle has side lengths $3s$, $4s$, $5s$ then there is a length $12 - 12s$ for the square, so its side length is $3 - 3s$ and $0 \leq s \leq 1$. The total area of the shapes is then $A(s) = \frac{1}{2} \cdot 3s \cdot 4s + (3 - 3s)^2 = 15s^2 - 18s + 9$. Since $A'(s) = 30s - 18$ is zero for $s = 3/5$, point list is $s = 0, 3/5, 1$. Since $A(0) = 9$, $A(3/5) = 18/5$, $A(1) = 6$, $\boxed{\text{minimum area is } 18/5}$ (at $s = 3/5$) and the $\boxed{\text{maximum area is } 9}$ (at $s = 1$).

Interlude!

Problem 6

Sand falls into a conical pile whose height is always 5/2 its radius. If the height of the sandpile is currently 5 meters and sand is being deposited onto the pile at a rate of π cubic meters per minute, how fast are the height and radius of the pile increasing? (Note: The volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

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Answer: We are given $r = \frac{2}{5}h$, so $V = \frac{1}{3}\pi r^2 h = \frac{4}{75}\pi h^3$.
Then $V'(t) = \frac{4}{25}\pi h^2 \cdot h'(t)$.

It is given that $h = 5\text{m}$ and $V' = \pi \frac{\text{m}^3}{\text{min}}$, so $h' = \dfrac{V'}{\frac{4}{25}\pi h^2} = \boxed{\dfrac{1}{4}\,\dfrac{\text{m}}{\text{min}}}$,

and then $r' = \frac{2}{5}h' = \boxed{\dfrac{1}{10}\,\dfrac{\text{m}}{\text{min}}}$.

Problem 7

For $f(x) = x^4 - 2x^2 + 3$, find and classify all critical numbers and find all intervals where f is increasing and where f is decreasing.

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Answer: Since $f' = 4x(x-1)(x+1)$, critical numbers are $x = -1, 0, 1$. Sign diagram is $f' : \underset{}{\ominus} \underset{-1}{|} \oplus \underset{0}{|} \ominus \underset{1}{|} \oplus$ so

$\boxed{\text{local minima at } x = \pm 1}$ and $\boxed{\text{local maximum at } x = 0}$, and $\boxed{\text{increasing on } (-1, 0), (1, \infty)}$, $\boxed{\text{decreasing on } (-\infty, -1), (0, 1)}$.

Interlude!

Problem 8

Evaluate $\int_0^{\pi/4} \sec(x)\tan(x)\,dx$.

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Answer:

Problem 8

Evaluate $\int_0^{\pi/4} \sec(x)\tan(x)\,dx$.

Answer: From basic integrals, $\int \sec(x)\tan(x)\,dx = \sec(x) + C$, so $\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \sec(x)\big|_{x=0}^{\pi/4} = \boxed{\sec(\frac{\pi}{4}) - \sec(0) = \sqrt{2} - 1}$.

Problem 9

Find $\int \sqrt{\sin^2 x + \cos^2 x}\, dx$.

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Answer:

Problem 9

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Answer: Note that $\sin^2 x + \cos^2 x = 1$, so the integral is just $\int 1\, dx = \boxed{x + C}$.

Problem 10

Compute $\int_0^{\pi/3} \sec^2(\star)\, d\star$.

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Answer:

Problem 10

Compute $\displaystyle\int_0^{\pi/3} \sec^2(\star)\, d\star$.

Answer: Since $\displaystyle\int \sec^2(\star)\, d\star = \tan(\star) + C$, we see

$$\int_0^{\pi/3} \sec^2(\star)\, d\star = \tan(\star)\Big|_{\star=0}^{\pi/3} = \boxed{\tan(\pi/3) - \tan(0) = \sqrt{3}}.$$

(don't be fooled by the variable $\star$: it behaves just like any other!)

Interlude!

Problem 11

Find the absolute minimum and maximum values of $f(x) = 2x + 4\sin(x)$ on $[0, \pi]$ and all places where they occur.

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Answer: We have $f'(x) = 2 + 4\cos(x)$ so critical numbers occur when $\cos(x) = -\frac{1}{2}$, and on $[0, \pi]$ the only such x is $x = \frac{2\pi}{3}$.

Including the endpoints, our point list is $x = 0, \frac{2\pi}{3}, \pi$. We compute $f(0) = 0$, $f(\frac{2\pi}{3}) = \frac{4\pi}{3} + 2\sqrt{3} \approx 7.653$, and $f(\pi) = 2\pi \approx 6.283$.

So the $\boxed{\text{max is } \dfrac{4\pi}{3} + 2\sqrt{3} \text{ at } x = \dfrac{2\pi}{3}}$ and the $\boxed{\text{min is } 0 \text{ at } x = 0}$.

Problem 12

Find $\lim_{x \to \infty} (1 + 2/x)^{3x}$.

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Answer:

Problem 12

Find $\lim_{x\to\infty}(1+2/x)^{3x}$.

Answer: This is a 1^∞ limit. Take the natural log to see $\ln L = \lim_{x\to\infty} \ln\left[(1+2/x)^{3x}\right] = \lim_{x\to\infty} 3x\ln(1+2/x) = \lim_{x\to\infty} \dfrac{3\ln(1+2/x)}{1/x}$.

Now apply L'Hôpital's rule:

$$\ln L = \lim_{x\to\infty} \frac{3(-2/x^2)/(1+2/x)}{-1/x^2} = \lim_{x\to\infty} \frac{3(-2)/(1+2/x)}{-1} = 6.$$

Therefore $\ln L = 6$ and so $L = \boxed{e^6}$.

Problem 13

A population of goats grows at a rate proportional to its current size. In 2010 the population is 500 and in 2020 the population is 1500. Find the population in 2030.

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Answer: The information says the population grows exponentially, so $P(t) = Ce^{kt}$ for some constants C and k, where we can take t to be years after 2010.

We are given $P(0) = 500$ and $P(10) = 1500$, so plugging in gives $C = 500$ and $Ce^{10k} = 1500$ so that $k = \ln(3)/10$.

The population in 2030 is $P(20) = \boxed{500e^{20\ln(3)/10} = 4500}$.

Problem 14

You cut squares of side length s in from each of the four corners of a rectangular piece of paper measuring 14 in by 30 in, and fold the resulting shape up into a box with no top. What value of s maximizes the volume?

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You cut squares of side length s in from each of the four corners of a rectangular piece of paper measuring 14 in by 30 in, and fold the resulting shape up into a box with no top. What value of s maximizes the volume?

Answer: Drawing the paper shows that the box will have a height of s, width $14 - 2s$, and length $30 - 2s$, so must have $0 \leq s \leq 7$. Volume is $V(s) = s(14 - 2s)(30 - 2s) = 4s^3 - 88s^2 + 420s$.

Then $V'(s) = 12s^2 - 176s + 420 = 4(3s - 35)(s - 3)$ so the only critical point in interval is at $s = 3$. Including endpoints, point list is $s = 0, 3, 7$. Since $V(0) = V(7) = 0$, the maximum is at $s = \boxed{3}$.

Problem 15

For $f(x) = xe^{-x^2/8}$, find and classify all critical numbers, find all inflection numbers, and find all intervals where f is increasing, decreasing, concave up, and concave down.

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Problem 15

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Answer: Since $f' = -\frac{1}{4}(x^2 - 4)e^{-x^2/8}$, the critical numbers are $x = \boxed{-2, 2}$. Sign diagram is $f' : \ominus \underset{-2}{|} \oplus \underset{2}{|} \ominus$ so $\boxed{\text{local min at } x = -2}$ and $\boxed{\text{local max at } x = 2}$, and $\boxed{\text{increasing on } (-2, 2)}$, $\boxed{\text{decreasing on } (-\infty, -2), (2, \infty)}$.

Also $f'' = \frac{1}{16}(x^2 - 12)e^{-x^2/8}$, the inflection numbers are $x = \boxed{-2\sqrt{3}, 2\sqrt{3}}$. Sign diagram is $f'' : \oplus \underset{-2\sqrt{3}}{|} \ominus \underset{2\sqrt{3}}{|} \oplus$ so $\boxed{\text{conc up on } (-\infty, -2\sqrt{3}), (2\sqrt{3}, \infty)}$, $\boxed{\text{conc down on } (-2, 2)}$.

End

Happy studying, and I will see you at the exam on Wednesday.