Math 1341 - Midterm 2 Review #1

November 14, 2019

Midterm 2 Topics:

- Related Rates
- Minimum and maximum values, critical points + classification
- Increasing and decreasing behavior, concavity
- Rolle's theorem + mean value theorem
- L'Hôpital's rule
- Antiderivatives
- Riemann sums + properties of definite integrals
- Fundamental theorem of calculus
- Evaluating definite and indefinite integrals

(Differentiating integrals and substitution are not on the midterm, though they are fair game for the final.)

Problem 1

Compute $\lim_{x \to 0} \dfrac{e^x - 1}{\sin(3x)}$.

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Answer:

Problem 1

Compute $\lim_{x \to 0} \frac{e^x - 1}{\sin(3x)}$.

Answer: Use L'Hôpital's rule to get $\lim_{x \to 0} \frac{e^x}{3\cos(3x)} = \boxed{\frac{1}{3}}$.

Problem 2

Find all intervals where $f(x) = x^5 + 5x^4 + 7$ is increasing, decreasing, concave up, concave down.

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Answer:

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Find all intervals where $f(x) = x^5 + 5x^4 + 7$ is increasing, decreasing, concave up, concave down.

Answer: Note $f' = 5x^3(x+4)$ so f has critical #s $x = 0, -4$. Using test points gives a sign diagram $f' : \underset{}{\oplus} \underset{-4}{|} \ominus \underset{0}{|} \oplus$, so f is

$\boxed{\text{incr on } (-\infty, -4) \text{ and } (0, \infty)}$ and $\boxed{\text{decr on } (0, 4)}$.

Likewise, $f'' = 20x^2(x+3)$ so f has inflection #s $x = 0, -3$. Using test points gives a sign diagram $f'' : \ominus \underset{-3}{|} \oplus \underset{0}{|} \oplus$, so f is

$\boxed{\text{conc up on } (-3, 0) \text{ and } (0, \infty)}$ and $\boxed{\text{conc down on } (-\infty, -3)}$.

Interlude!

Problem 3

The sum of three positive numbers is 12 and two of them are equal. Find the largest possible product.

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Answer: Suppose the numbers are x, x, y. Then $2x + y = 12$ meaning that $y = 12 - 2x$, and so the product is $p(x) = x^2(12 - 2x) = 12x^2 - 2x^3$. Since we must have $0 < x < 6$ we look for critical #s in this range.

Since $p'(x) = 24x - 6x^2 = 6x(4 - x)$, the only critical # of p in that range is $x = 4$, which is local max via the sign diagram for p'. This means the maximum product is $p(4) = \boxed{64}$.

Problem 4

Evaluate $\int (x^2 + 1)^2 \, dx$.

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Answer:

Problem 4

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Answer: Expand the integrand:

$$\int (x^2+1)^2\,dx = \int (x^4+2x^2+1)\,dx = \boxed{\frac{1}{5}x^5+\frac{2}{3}x^3+x+C}.$$

Interlude!

Problem 5

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Answer:

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Find the absolute minimum and maximum of $f(x) = x + 16/x$ on the interval $[1, 8]$, and all places where they occur.

Answer: Since $f'(x) = 1 - \dfrac{16}{x^2}$, the critical #s occur when $x = -4, 4$. So the only critical # in the interval is at $x = 4$.

So, including the endpoints, our point list is $x = 1, 4, 8$.

Since $f(1) = 17$, $f(4) = 8$, and $f(8) = 10$, the $\boxed{\text{minimum is 8 at } x = 4}$ and the $\boxed{\text{maximum is 17 at } x = 1}$.

Problem 6

The volume of a cylindrical block of ice is $V = \pi r^2 h$. If the radius r is currently 10cm and decreasing at $1\frac{\text{cm}}{\text{min}}$ and the height h is currently 20cm and decreasing at $3\frac{\text{cm}}{\text{min}}$, how fast is the volume decreasing?

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Answer: Taking the derivative of the volume formula gives $V'(t) = 2\pi rh \cdot r'(t) + \pi r^2 \cdot h'(t)$. The given information says $r = 10\text{cm}$, $r' = -1 \frac{\text{cm}}{\text{min}}$, $h = 20\text{cm}$, $h' = -3 \frac{\text{cm}}{\text{min}}$.

Plugging in gives $V' = \boxed{-700\pi \frac{\text{cm}^3}{\text{min}}}$.

Problem 7

Evaluate $\int_2^2 \sqrt{3e^{2x} + \sin(x)}\, dx$.

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Answer:

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Answer: The top and bottom limits are the same so the integral is $\boxed{0}$, regardless of the function.

Interlude!

Problem 8

Evaluate $\displaystyle\int_1^e \frac{x^2 - x + 1}{x}\,dx$.

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Answer:

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Answer: Distributing the fraction gives

$$\int_1^e \frac{x^2 - x + 1}{x}\,dx = \int_1^e \left(x - 1 + \frac{1}{x}\right)\,dx = \left(\frac{1}{2}x^2 - x + \ln(x)\right)\Bigg|_{x=1}^{e} = \boxed{\frac{1}{2}(e^2 - 2e + 3)}.$$

Problem 9

Find all critical numbers of $f(x) = f(x) = x^3 + 9x^2 - 21x + 2$ and classify them as local minima, local maxima, or neither.

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Answer: Note that $f'(x) = 3x^2 + 18x - 21 = 3(x-1)(x+7)$ so the critical numbers are $x = \boxed{-7, 1}$.

Using test points gives a sign diagram $f' : \underset{-7}{\oplus \mid} \underset{1}{\ominus \mid} \oplus$, so there is a $\boxed{\text{local maximum at } x = -7}$ and a $\boxed{\text{local minimum at } x = 1}$.

Interlude!

Problem 10

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Answer:

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Answer: Use L'Hôpital's rule: $L = \lim_{x \to 0} \frac{2x \cos(x^2)}{\sin(x)}$. This is still indeterminate so use it again: $L = \lim_{x \to 0} \frac{2\cos(x^2) - 4x^2 \sin(x^2)}{\cos(x)}$ which then evaluates to $\boxed{2}$ at $x = 0$.

Problem 11

Find $f(x)$ if $f''(x) = 12x^2 - e^x + \sin(x)$, where $f'(0) = 1$ and $f(0) = 2$.

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Answer: Take the antiderivative to get
$f'(x) = 4x^3 - e^x - \cos(x) + C$. Then $f'(0) = 1$ says
$f'(0) = -1 - 1 + C$ so $C = 3$, and $f'(x) = 4x^3 - e^x - \cos(x) + 3$.

Take the antiderivative again to see
$f(x) = x^4 - e^x - \sin(x) + 3x + D$. Then $f(0) = 2$ says
$f(0) = -1 + D$ so $D = 3$. So $f(x) = \boxed{x^4 - e^x - \sin(x) + 3x + 3}$.

Interlude!

Problem 12

Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin(2x)}{\cos^2(x)}\,dx$.

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Answer:

Problem 12

Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin(2x)}{\cos^2(x)}\,dx$.

Answer: Here we need to use the trig identity $\sin(2x) = 2\sin(x)\cos(x)$. This gives $I =$

$$\int_{\pi/6}^{\pi/3} \frac{2\sin(x)\cos(x)}{\cos^2(x)}\,dx = \int_{\pi/6}^{\pi/3} \frac{2\sin(x)}{\cos(x)}\,dx = \int_{\pi/6}^{\pi/3} 2\tan(x)\,dx.$$

Then $\int_{\pi/6}^{\pi/3} 2\tan(x)\,dx = -2\ln(\cos(x))\big|_{x=\pi/6}^{\pi/3} = \boxed{\ln 3}$.

Problem 13

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Answer: The interval width is $\dfrac{1-0}{5} = \dfrac{1}{5}$ so the intervals are $[0, \frac{1}{5}]$, $[\frac{1}{5}, \frac{2}{5}]$, $[\frac{2}{5}, \frac{3}{5}]$, $[\frac{3}{5}, \frac{4}{5}]$, $[\frac{4}{5}, 1]$.

So the Riemann sum is $[f(0) + f(1/5) + f(2/5) + f(3/5) + f(4/5)] \cdot \frac{1}{5}$, which evaluates to $[0^2 + 1/25 + 4/25 + 9/25 + 16/25] \cdot \frac{1}{5} = \frac{30}{125} = \boxed{0.24}$.

Problem 14

Find the absolute minimum and maximum of $f(x) = x^2e^x$ on the interval $[-3, 1]$ and all places where they occur.

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Answer: We have $f'(x) = 2xe^x + x^2e^x = x(x+2)e^x$ so the critical numbers are where f' is zero, which occurs for $x = -2, 0$. Including the endpoints, our point list is $x = -3, -2, 0, 1$.

We have $f(-3) = 9/e^3$, $f(-2) = 4/e^2$, $f(0) = 0$, $f(1) = e$. The $\boxed{\text{minimum is } 0 \text{ at } x = 0}$ and the $\boxed{\text{maximum is } e \text{ at } x = 1}$.

Problem 15

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Answer: Since f is continuous with $f(-1) = -1$ and $f(0) = 1$, by the Intermediate Value Theorem, f has a real root in $(-1, 0)$.

Also, since $f' = 3x^2 + 3$ is never zero, f cannot have 2 roots since by Rolle's theorem, f' would be zero somewhere between them.

So f has exactly 1 real root.

End

Enjoy WeBWorK #10, and I will see you on Monday for more review!