Math 1341 - Midterm 1 Review # 2

October 16, 2019

General List of Exam Topics

- Limits (finite limits, infinite limits, limits at infinity)
- Continuity
- Limit definition of derivative, differentiability
- Computing derivatives (product, quotient, chain rule)
- Derivatives of inverse functions
- Logarithmic differentiation
- Implicit differentiation
- Parametric differentiation (velocity, speed, acceleration)
- Tangent lines and rates of change
- Linearization and linear approximation

Problem 1

Calculate the average rate of change of $f(x) = 2x^2 + 2$ on the interval $[1, 3]$.

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Answer:

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Answer: The average rate of change is
$$\frac{f(3) - f(1)}{3 - 1} = \frac{20 - 4}{3 - 1} = \boxed{8}.$$

Problem 2

Use the limit definition of the derivative to find $s'(1)$ for $s(t) = \sqrt{3t+1}$.

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Answer: By definition, this is

$$
\begin{aligned}
s'(1) &= \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(1+h)+1} - 2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3h+4} - 2}{h} \cdot \frac{\sqrt{3h+4}+2}{\sqrt{3h+4}+2} \\
&= \lim_{h\to 0} \frac{3h}{h \cdot (\sqrt{3h+4}+2} = \frac{3}{4}.
\end{aligned}
$$

Problem 3

Calculate $\lim_{x \to 1} \frac{1}{(x-1)^6}$.

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Answer:

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Calculate $\lim_{x \to 1} \frac{1}{(x-1)^6}$.

Answer: As $x \to 1-$, the denominator approaches zero and is positive, while the numerator is positive, so the limit as $x \to 1-$ is $+\infty$. As $x \to 1+$, the denominator approaches zero and is positive, while the numerator is positive, so the limit as $x \to 1+$ is $+\infty$. These values are equal so the overal limit is $\boxed{+\infty}$.

Problem 4

Find $f''(x)$ if $f(x) = \tan^{-1}(x)$.

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Answer:

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Find $f''(x)$ if $f(x) = \tan^{-1}(x)$.

Answer: First, $f'(x) = \dfrac{1}{x^2+1}$. Then by the quotient rule, we see $f''(x) = \boxed{-\dfrac{2x}{(x^2+1)^2}}$.

Interlude!

Problem 5

Use logarithmic differentiation to find the derivative of
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Answer: First take the natural logarithm and simplify, yielding $\ln(f) = \frac{1}{2}\cos(x) \cdot \ln(\sin(x))$. Now take the derivative of both sides, which gives $\frac{f'}{f} = \left[-\frac{1}{2}\sin(x)\ln(\sin(x)) + \frac{1}{2}\cos(x) \cdot \frac{\cos(x)}{\sin(x)}\right]$.
Finally, solving for f' yields

$$f'(x) = \boxed{\sqrt{(\sin x)^{\cos x}}\left[-\frac{1}{2}\sin(x)\ln(\sin(x)) + \frac{1}{2}\cos(x) \cdot \frac{\cos(x)}{\sin(x)}\right]}.$$

Problem 6

Consider the implicit curve $x^2y + x^5y^6 = 2$, which defines y implicitly as a function of x. Find an equation for the line tangent to the curve at the point $(x, y) = (1, 1)$.

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Answer: First use implicit differentiation to find the slope of the tangent line $dy/dx = y'$: we get
$2xy + x^2y' + 5x^4y^6 + x^5 \cdot 6y^5y' = 0$, so solving yields
$y' = -\dfrac{2xy + 5x^4y^6}{x^2 + 6x^5y^5}$. Then the slope is $\dfrac{dy}{dx}$ at $(x, y) = (1, 1)$, which is -1. Thus the equation is $\boxed{y - 1 = -(x - 1)}$, or equivalently, $\boxed{y = -x + 2}$.

Problem 7

Calculate $\dfrac{d}{dt}\left[\sqrt{\ln(\sin(t))}\right]$.

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Answer: By the chain rule (repeatedly), we obtain the derivative $\boxed{\frac{1}{2}[\ln(\sin(t))]^{-1/2} \cdot \cos(t) \cdot \frac{1}{\sin(t)}}$.

Problem 8

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Answer: By definition, the linearization of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a) \cdot (x - a)$. Since $f(1) = 10$ and $f'(1) = 30$, the linearization is $L(x) = \boxed{10 + 30(x - 1)}$.

Interlude!

Problem 9

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Answer: The first derivative is $f'(x) = 3x^2 \cdot 2^x + x^3 \cdot 2^x \ln(2)$ by the product rule. Then $f'(2) = \boxed{48 + 32\ln(2)}$.

Problem 10

The function $f(x) = 4x + \sin(3x)$ is one-to-one, so it has an inverse function $g(x)$. Find $g'(4\pi)$.

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Answer: This is a "differentiating an inverse function" problem. The formula to use here is $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$. Since $f(\pi) = 4\pi$ we see $g(4\pi) = f^{-1}(4\pi) = \pi$. Note $f'(x) = 4 + 3\cos(3x)$. Then by the formula this gives

$$g'(4\pi) = \frac{1}{f'(f^{-1}(4\pi))} = \frac{1}{f'(\pi)} = \frac{1}{4 + 3\cos(3\pi)} = \boxed{1}.$$

Problem 11a

A particle moves through the plane so that at time t seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find the particle's velocity, speed, and acceleration at time t.

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Answer: The velocity is $(x', y') = \boxed{(6e^{2t}, 6e^{6t})}$. The speed is $\sqrt{[x'(t)]^2 + [y'(t)]^2} = \boxed{\sqrt{(6e^{2t})^2 + (6e^{6t})^2}}$. And the acceleration is $(x'', y'') = \boxed{(12e^{2t}, 36e^{6t})}$.

Problem 11b

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Problem 11b

A particle moves through the plane so that at time t seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find an equation for the tangent line to the particle's path at time $t = 0$.

Answer: First, at time $t = 0$, the particle's position is $(x(0), y(0)) = (3, 1)$. We also need the slope of the tangent line, which at time t is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6e^{6t}}{6e^{2t}} = e^{4t}$. So at time $t = 0$ the slope is 1, and therefore the equation of the tangent line is $\boxed{y - 1 = 1(x - 3)}$.

Interlude!

Problem 12

Suppose that $f(1) = 5$, $f'(1) = 6$, $f(5) = 5$, $f'(5) = 2$, $g(1) = 5$, and $g'(1) = 8$. Find the derivative of $f(x)/g(x)$ at $x = 1$.

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Answer:

Problem 12

Suppose that $f(1) = 5$, $f'(1) = 6$, $f(5) = 5$, $f'(5) = 2$, $g(1) = 5$, and $g'(1) = 8$. Find the derivative of $f(x)/g(x)$ at $x = 1$.

Answer: By the quotient rule, this is

$$\frac{f'(1)g(1) - f(1)g'(1)}{[g(1)]^2} = \boxed{-\frac{2}{5}}.$$

Problem 13

Using an appropriate linear approximation, estimate the value of $\sqrt[5]{32.08}$.

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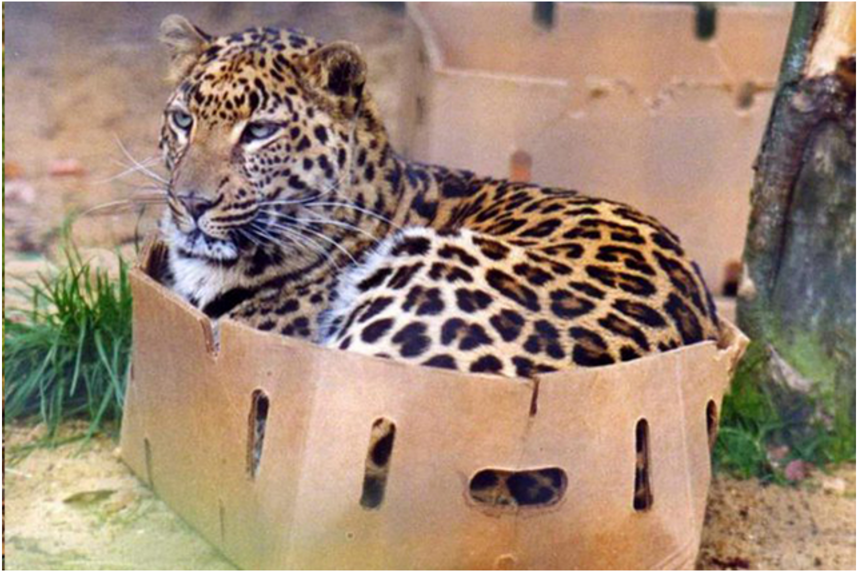
Answer: We use the linearization of $f(x) = \sqrt[5]{x} = x^{1/5}$ at $a = 32$, since we can easily evaluate $f(32)$. The linearization is $L(x) = f(a) + f'(a) \cdot (x - a)$, so since $f'(x) = \frac{1}{5}x^{-4/5}$ we compute $f(32) = 2$ and $f'(32) = \frac{1}{80}$. Then the desired estimate is $L(32.08) = 2 + \frac{1}{80}(32.08 - 32) = \boxed{2.001}$.

Interlude!



End of Review

Good luck with your studying, and I'll see you tomorrow for the exam!