Math 1341 - Midterm 1 Review

October 10, 2019

Midterm 1 Topics:

- Limits (finite limits, infinite limits, limits at infinity)
- Continuity
- Limit definition of derivative, differentiability
- Computing derivatives (product, quotient, chain rule)
- Derivatives of inverse functions
- Logarithmic differentiation
- Implicit differentiation
- Parametric differentiation (velocity, speed, acceleration)
- Tangent lines and rates of change
- Linearization and linear approximation

(Related rates are on midterm 2!)

Problem 1

Find the derivative of $(2x^3 + 6)^{11}$.

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Answer:

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Answer: Use the Chain Rule to get $\boxed{11 \cdot (2x^3 + 6)^{10} \cdot 6x^2}$.

Problem 2

Calculate $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x^2 - 4}$.

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Answer:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(x-1)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x-1}{x+2} = \boxed{\frac{1}{4}}.$$

Problem 3

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Answer: The velocity is $(x', y') = (6t, 3t^2 - 1)$ meters per second. Then the acceleration is
$(x'', y'') = (6, 6t)$ meters per second squared, so at time $t = 2$ seconds, it is $\boxed{(6, 12) \text{ meters per second squared}}$. (Yes, the units are required!)

Problem 4

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Answer: The line passes through $(0, y(0)) = (0, 1)$ and has slope $y'(0) = 4$, so it is $\boxed{y - 1 = 4(x - 0)}$, alternatively written as $\boxed{y = 4x + 1}$. [Note that $y'(x) = 2(x+1) + 2\cos(2x)$.]

Interlude!

Problem 5

Find the derivative of $\sin^2(5x + \cos(5x))$.

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Answer: $\boxed{2\sin(5x + \cos(5x)) \cdot \cos(5x + \cos(5x)) \cdot [5 - 5\sin(5x)]}$.

Problem 6

Use an appropriate linearization to estimate the value of $\sqrt{3.92}$.
(No calculators!)

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Answer: The linearization of $\sqrt{x}$ near $x = 4$ is $L(x) = 2 + \frac{1}{4}(x - 4)$. So $\sqrt{3.92} \approx \boxed{2 + \frac{1}{4} \cdot (3.92 - 4) = 1.98}$.

Problem 7

Find $\lim_{x \to 1} \frac{x-3}{(x-1)^3}$.

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Answer:

Problem 7

Find $\lim_{x \to 1} \frac{x-3}{(x-1)^3}$.

Answer: The left limit is ∞ (numerator is negative, denominator is small negative) while the right limit is $-\infty$ (numerator is negative, denominator is small positive) so the two-sided limit $\boxed{\text{does not exist}}$.

Problem 8

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Answer: This is an "inverse function differentiation formula" problem. First, we observe that $f(0) = 2$, and $f'(x) = 9x^8 + 3 - 2\sin(x)$. So we have $g(2) = 0$, and then by the formula, we know that $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(0)} = \boxed{\dfrac{1}{3}}$.

Interlude!

Problem 9

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Answer:

$$\lim_{x\to\infty} \frac{\sqrt{3x^3 - 2x}}{x^2 + 2x} = \lim_{x\to\infty} \frac{x^{3/2}\sqrt{3 - 2/x^3}}{x^2(1 + 2/x)} = \lim_{x\to\infty} x^{-1/2} \cdot \frac{\sqrt{3 - 2/x^3}}{1 + 2/x},$$

which is equal to $\boxed{0}$.

Problem 10

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Answer: Use logarithmic differentiation:
$\ln(f) = 2\sin(x) \cdot \ln(4x+1)$, so taking the derivative of both sides yields $\frac{f'}{f} = 2\cos(x) \cdot \ln(4x+1) + 2\sin(x) \cdot \frac{4}{4x+1}$. Therefore,

$$f' = \boxed{(4x+1)^{2\sin(x)} \cdot \left[2\cos(x) \cdot \ln(4x+1) + 2\sin(x) \cdot \frac{4}{4x+1}\right]}.$$

Problem 11

Suppose that y is defined as an implicit function of x by the relation $ye^{2x} + y^3 = 2$. What is $\frac{dy}{dx}$ when $x = 0$ and $y = 2$?

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Answer: Differentiating implicitly yields
$\frac{dy}{dx} \cdot e^{2x} + ye^{2x} \cdot 2 + 3y^2 \cdot \frac{dy}{dx} = 0$. Solving yields
$\frac{dy}{dx} = -\frac{ye^{2x} \cdot 2}{e^{2x} + 3y^2}$, which at $(x, y) = (0, 2)$ is equal to $-\boxed{4/13}$.

Interlude!

Problem 12

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Problem 12

True or False: Because the derivative of a^x is $a^x \ln(a)$, the derivative of x^x is $x^x \ln(x)$.

Answer: False! : The formula for the derivative of a^x only holds when a is a constant. If we wanted to find the correct derivative of $f(x) = x^x$, we could observe that $\ln(f(x)) = x \ln(x)$, and then take the derivative of both sides to see that $\dfrac{f'(x)}{f(x)} = \ln(x) + x \cdot \dfrac{1}{x}$, so that $f'(x) = x^x(\ln(x) + 1)$.

Problem ∞

Find the 2019th derivative of $f(x) = \dfrac{\tan^{-1}(x^3+2)}{\sin(\ln(x))}$.

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Answer: Just kidding, that would be horrible. Enjoy the long weekend!